Lecture 3 – Holographic Universe and inflation

AdS/CFT and braneworld holography
Holographic cosmology
k-essence inflation

Basic idea

A braneworld can be located at the boundary of a 5-dim asymptotically Anti de Sitter space (AdS_5). In this case the cosmic evolution of the braneworld will be governed by matter on the brane in addition to the conformal fluid dual to the gravity in the bulk.

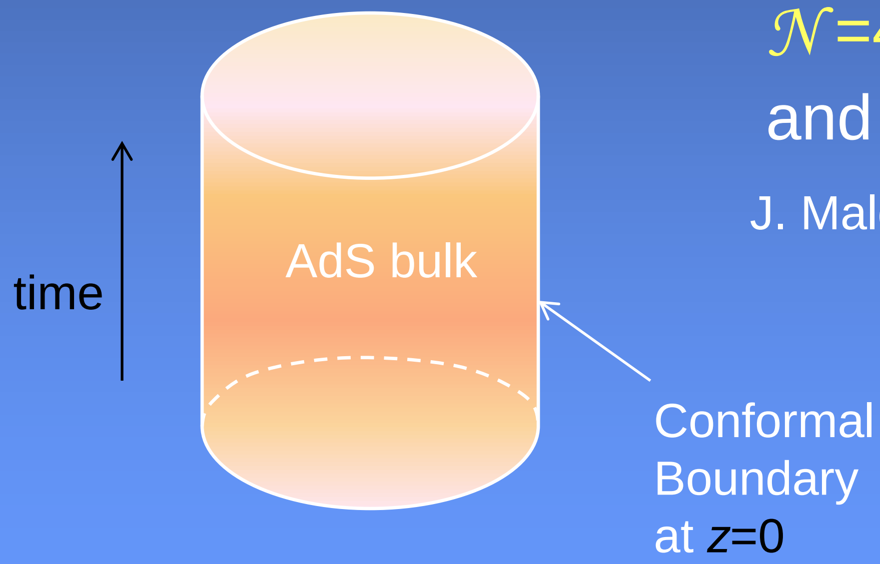
As the stress tensor of the conformal fluid is determined by the geometry of the bulk we expect a deviation from the standard FRW cosmology on the brane.

AdS/CFT and braneworld holography

AdS/CFT correspondence is a holographic duality between gravity in d+1-dim space-time and quantum CFT on the d-dim boundary. Original formulation stems from string theory:

Equivalence of 3+1-dim
$\mathcal{N}$=4 Supersymmetric YM Theory
and string theory in $AdS_5 \times S_5$

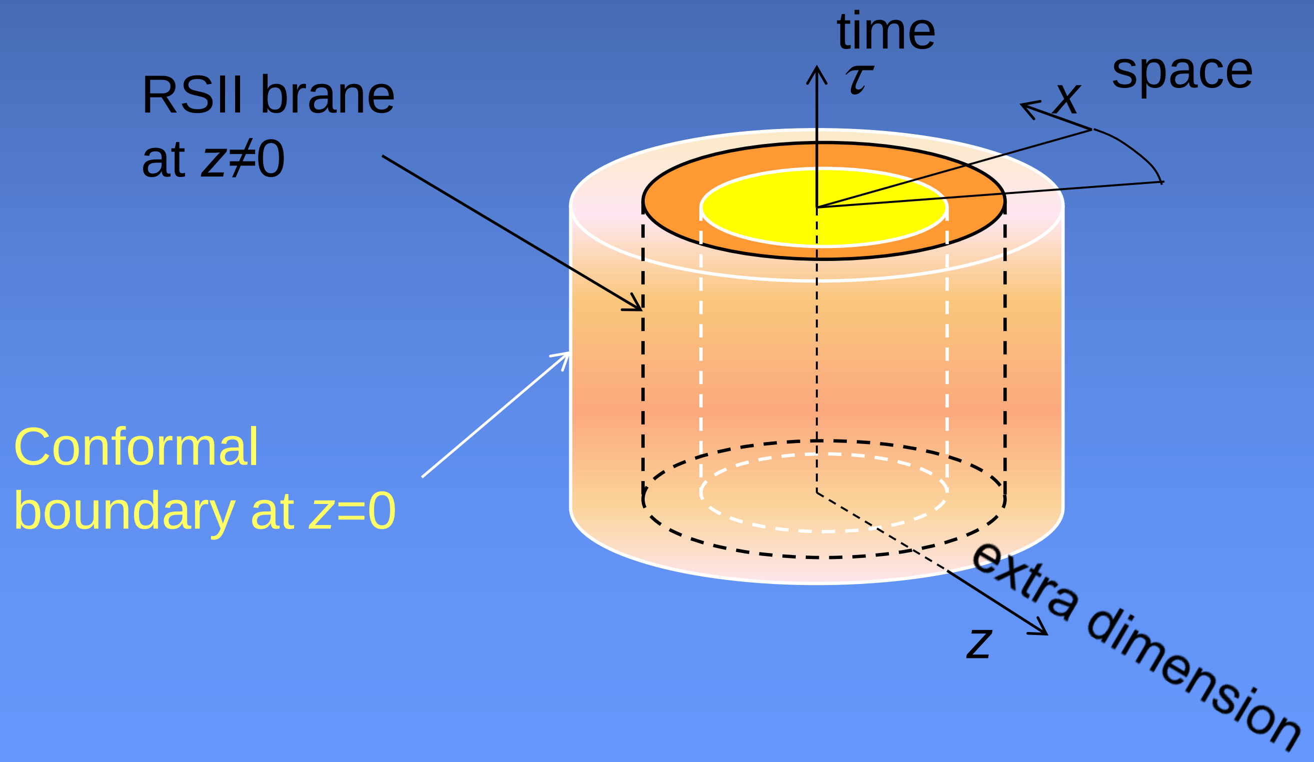
J. Maldacena, Adv. Theor. Math. Phys. **2** (1998)



Examples of CFT:
Maxwell electrodynamics,
Massless Maxwell-Dirac ED
Massless φ^4 scalar field theory

In the second Randall-Sundrum (RS II) model a 3-brane is located at a finite distance from the boundary of AdS_5 .

Holographic braneworld is a 3-brane located at the boundary of the asymptotic AdS_5. The cosmology is governed by matter on the brane in addition to the boundary CFT



Why AdS?

Anti de Sitter space is a maximally symmetric solution to Einstein's equations with negative cosmological constant.
In 4+1 dimensions the symmetry group is AdS$_5$≡ **SO(4,2)**

The bulk metric may be represented by (Fefferman-Graham coordinates)

$$ds^2_{(5)} = G_{ab} dX^a dX^b = \frac{\ell^2}{z^2}(g_{\mu\nu} dx^\mu dx^\nu - dz^2)$$

So there is a boundary at $z = 0$. A correspondence between gravity in the bulk and the conformal field theory (CFT) on the boundary of AdS may be expected because the 3+1 boundary conformal field theory is invariant under conformal transformations: Poincare + dilatations + special conformal transformation = **conformal group ≡ SO(4,2)**

It is sometimes convenient to represented the metric in Gaussian normal coordinates which we have previously employed for the RSII model

$$ds^2_{(5)} = e^{-2ky} g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2$$

In these coordinates, the boundary is at $y = -\infty$

Based on:

N.B. , D.D. Dimitrijevic, G.S. Djordjevic, M. Milosevic, and M. Stojanovic, *Tachyon inflation in the holographic braneworld*, JCAP08(2019)034, arXiv:1809.0721,
N.B., *Holographic cosmology and tachyon inflation* , Int.J.Mod.Phys. **A33** (2018), arXiv:1808.08146
N.B., *Randall-Sundrum versus holographic cosmology*, Phys. Rev. D 93, (2016) arXiv:1511.07323
N.R. Bertini, N.B., and D.C. Rodrigues, *Primordial perturbations and inflation in holographic cosmology*, Phys. Rev. **D 102**, 123505 (2020); Phys. Rev. D 105, 129901 (2022) (Erratum) arXiv:2007.02332

and related earlier works

P.S. Apostolopoulos, G. Siopsis and N. Tetradis, Phys. Rev. Lett. **102** (2009), arXiv:0809.3505
P. Brax and R. Peschanski, Acta Phys. Polon. **B 41** (2010) arXiv:1006.3054

Consider the matter part of a 5-dim bulk action in asymptotic AdS_5 background

$$S_{(5)}[\Phi] = \int d^5x\sqrt{G}\,\mathcal{L}_{(5)}(\Phi, G_{ab})$$

The bulk field Φ is completely determined by its field equations obtained from the variational principle

$$\frac{\delta S_{(5)}}{\delta \Phi} = 0$$

given the boundary value $\varphi(x) \equiv \Phi(z=0,x)$ and the induced metric on the boundary $h_{\mu\nu}$.

Using the solution $\Phi = \Phi[\varphi, h]$ we can define a functional

$$S[\varphi, h] = S^{\text{shell}}\left[\Phi[\varphi, h]\right]$$

where $S^{\text{shell}}\left[\Phi[\varphi, h]\right]$ is the action **on-shell** bulk action, i.e., the action in which the fields are solutions to the field equations given their boundary values. The on-shell bulk action is still subject to the variation of the boundary values.

AdS/CFT conjecture: The action $S[\varphi,h]$ can be identified with the generating functional of a conformal (quantum) field theory on the boundary

$$S[\varphi,h] \equiv \ln \int d\psi \exp\left\{-\int d^4x\sqrt{-h}\left[\mathcal{L}^{\mathrm{CFT}}(\psi(x)) - O(\psi(x))\varphi(x)\right]\right\}$$

where the boundary fields serve as sources for CFT operators

$\mathcal{L}^{\mathrm{CFT}}(\psi)$ – conformal field theory Lagrangian

$O(\psi)$ – operators of dimension Δ

In this way the CFT correlation functions can be calculated as functional derivatives of the on-shell bulk action, e.g.,

$$\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} = \langle O(\psi(x))O(\psi(y))\rangle - \langle O(\psi(x))\rangle\langle O(\psi(y))\rangle$$

Consider next a bulk action with **only gravity** in the bulk

$$S = \frac{1}{8\pi G_5}\int d^5x\sqrt{G}\left(-\frac{R^{(5)}}{2} - \Lambda_5\right)$$

Given induced metric $h_{\mu\nu}$ on the boundary the geometry is completely determined by the field equations obtained from the variation principle

$$\frac{\delta S}{\delta G_{ab}} = 0$$

yielding a solution $G_{ab}[h]$. Using this we define a functional

$$S[h] = S^{\text{shell}}[G_{ab}[h]]$$

where $S^{\text{shell}}\,[G_{ab}[h]]$ is the on-shell bulk action

AdS/CFT conjecture: the action $S[h]$ can be identified with the generating functional of a conformal field theory (CFT) on the boundary.

The induced metric $h_{\mu\nu}$ serves as the source for the **energy-momentum tensor** of the dual CFT so that its vacuum expectation value is obtained from the on shell classical action

$$\langle T_{\mu\nu}^{\mathrm{CFT}} \rangle = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h^{\mu\nu}}$$

Holographic renormalization

The on-shell bulk action is IR divergent and must be regularized and renormalized. The asymptotically AdS metric in the Fefferman-Graham form is

$$ds^2_{(5)} = G_{ab}dx^a dx^b = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2) \qquad \mu,\nu = 0,1,2,3$$

where the length scale ℓ is the AdS curvature radius.

Near z=0 the four-tensor $g_{\mu\nu}$ can be expanded as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots \qquad (38)(3.1)$$

Explicit expressions for $g^{(2n)}_{\mu\nu}$ in terms of arbitrary boundary metric $g^{(0)}_{\mu\nu}$ can be found in

S. de Haro, S.N. Solodukhin, K. Skenderis, Comm. Math. Phys. **217** (2001)

In particular, we will need $g_{\mu\nu}^{(2)}$ and $g_{\mu\nu}^{(4)}$

$$g_{\mu\nu}^{(2)} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}^{(0)}\right) \tag{3.2}$$

$$g_{\mu\nu}^{(4)} = \frac{1}{4} g_{\mu\rho}^{(2)} g^{(0)\rho\sigma} g_{\sigma\nu}^{(2)} + \tilde{g}_{\mu\nu}^{(4)} \tag{3.3}$$

where the tensor $\tilde{g}_{\mu\nu}^{(4)}$ depends on the boundary metric $g_{\mu\nu}^{(0)}$ and vanishes if this metric is conformally flat, such as the FRW metric. In the following we will ignore the contribution of $\tilde{g}_{\mu\nu}^{(4)}$ as we will focus on the FRW cosmology on the boundary.

In the RSII model by introducing the boundary in AdS_5 at $z = z_{br}$ instead of $z = 0$, the model is conjectured to be dual to a cutoff CFT coupled to gravity, with $z = z_{br}$ providing the IR cutoff (corresponding to the UV cutoff of the boundary CFT)

So, we regularize the action by placing a brane (an RSII brane) near the AdS boundary, i.e., at $z = \varepsilon\ell, \quad \varepsilon << 1$, so that the induced metric on the brane is

$$h_{\mu\nu} = \frac{1}{\varepsilon^2}\left(g^{(0)}_{\mu\nu} + \varepsilon^2\ell^2 g^{(2)}_{\mu\nu} + \cdots\right)$$

The bulk splits in two regions: $0 \leq z \leq \varepsilon\ell$, and $\varepsilon\ell \leq z < \infty$. We can either discard the region $0 \leq z \leq \varepsilon\ell$ (one-sided regularization) or invoke the Z_2 symmetry (as in the original RSII model) and identify two regions (two-sided regularization). As before, we use one-sided regularization.

The regularized bulk action is given by

$$S^{\mathrm{reg}}[h] = \frac{1}{8\pi G_5} \int\limits_{z \geq \varepsilon \ell} d^5x \sqrt{G} \left(-\frac{R^{(5)}}{2} - \Lambda_5 \right) + S_{GH}[h] + S_{\mathrm{br}}[h]$$

$h_{\mu\nu}$ – induced metric on the brane

$S_{\mathrm{GH}}[h]$ – Gibbons-Hawking boundary term

$S_{\mathrm{br}}[h] = -\sigma \int d^4x \sqrt{-h} + \int d^4x \sqrt{-h}\, \mathcal{L}_{\mathrm{matt}}$ – brane action

The equations of motion on the brane are obtained by demanding that the variation with respect to the induced metric $h^{\mu\nu}$ of the regularized on shell bulk action (RSII action) vanishes, i.e.,

$$\delta S^{\mathrm{reg}}[h] = 0$$

Next we renormalize the boundary action. The renormalized boundary action is obtained by adding counter-terms and taking the limit ε→0

$$S^{\text{reg}}[h] = \lim_{\varepsilon\to 0}(S^{\text{ren}}[h] + S_1[h] + S_2[h] + S_3[h])$$

The necessary counter-terms are

$$S_1[h] = -\frac{6}{16\pi G_5 \ell}\int d^4x\sqrt{-h},$$

$$S_2[h] = -\frac{\ell}{16\pi G_5}\int d^4x\sqrt{-h}\left(-\frac{R[h]}{2}\right),$$

$$S_3[h] = -\frac{\ell^3}{16\pi G_5}\int d^4x\sqrt{-h}\frac{\log\epsilon}{4}\left(R^{\mu\nu}[h]R_{\mu\nu}[h] - \frac{1}{3}R^2[h]\right)$$

S.W. Hawking, T. Hertog, and H.S. Reall, Phys. Rev. D **62** (2000)

Now we demand that the variation with respect to the induced metric $h^{\mu\nu}$ of the regularized on shell bulk action (RSII action) vanishes, i.e.,

$$\delta S^{\mathrm{reg}}[h] = 0$$

This may be expressed as

matter on the brane

$$\delta\left[S^{\mathrm{ren}} - S_3 - \left(\sigma - \frac{3}{8\pi\ell G_5}\right)\int d^4x\sqrt{-h} + \int d^4x\sqrt{-h}\,\mathcal{L}_{\mathrm{matt}} + \frac{\ell}{16\pi G_5}\int d^4x\sqrt{-h}\,\frac{R}{2}\right] = 0.$$

cosmological Constant term

Einstein Hilbert term

AdS/CFT prescription

$$\delta(S^{\mathrm{ren}} - S_3) = \frac{1}{2}\int d^4x\sqrt{-h}\left\langle T^{\mathrm{CFT}}_{\mu\nu}\right\rangle \delta h^{\mu\nu}$$

The variation of the scheme-dependent S_3 combined with S^{ren} yields

$$\frac{2}{\sqrt{-h}}\frac{\delta(S^{\mathrm{ren}}-S_3)}{\delta h^{\mu\nu}} = \left\langle T_{\mu\nu}^{\mathrm{CFT}} \right\rangle$$

Then, the variation of the total action $S^{\mathrm{reg}}[h]$ yields Einstein's equations on the boundary

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}^{(0)} = 8\pi G_{\mathrm{N}}\left(\left\langle T_{\mu\nu}^{\mathrm{CFT}}\right\rangle + T_{\mu\nu}\ \right)$$

metric on the boundary

matter on the boundary

$$T_{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta S^{\mathrm{matt}}}{\delta h^{\mu\nu}}\bigg|_{g_{\mu\nu}^{(0)}} = 2\frac{\partial \mathcal{L}^{\mathrm{matt}}}{\partial h^{\mu\nu}}\bigg|_{g_{\mu\nu}^{(0)}} - \mathcal{L}^{\mathrm{matt}}\ g_{\mu\nu}^{(0)}$$

The vacuum expectation value of the conformal stress tensor is calculated by de Haro, Solodukhin, and Skenderis :

$$\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle = -\frac{\ell^3}{4\pi G_5}\left\{g^{(4)}_{\mu\nu} - \frac{1}{8}\left[(\mathrm{Tr}g^{(2)})^2 - \mathrm{Tr}(g^{(2)})^2\right]g^{(0)}_{\mu\nu} - \frac{1}{2}(g^{(2)})^2_{\mu\nu} + \frac{1}{4}\mathrm{Tr}g^{(2)}g^{(2)}_{\mu\nu}\right\}$$

where $g^{(2n)}_{\mu\nu}$ are the coefficients given by (3.2) and (3.3) that appear in the expansion (3.1) of the bulk metric

Using (3.2) and (3.3) for a conformally flat boundary metric, we find

$$\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle = -\frac{\ell^2}{32\pi G_{\mathrm{N}}}\left[\frac{2}{3}RR_{\mu\nu} - R_{\mu\rho}R^{\rho}{}_{\nu} + \frac{1}{4}\left(2R_{\alpha\beta}R^{\alpha\beta} - R^2\right)g^{(0)}_{\mu\nu}\right]$$

This is an explicit realization of the AdS/CFT correspondence:
the vacuum expectation value of a boundary CFT operator is obtained in terms of geometrical quantities of the bulk.

In this way we obtain the holographic Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g^{(0)}_{\mu\nu} + \overbrace{\frac{\ell^2}{4}\left[\frac{2}{3} R R_{\mu\nu} - R_{\mu\rho} R^{\rho}{}_{\nu} + \frac{1}{4}\left(2 R_{\alpha\beta} R^{\alpha\beta} - R^2\right) g^{(0)}_{\mu\nu}\right]}^{\text{modification}}$$

$$= 8\pi G_{\mathrm{N}} (T_{\mu\nu} + t_{\mu\nu}) \qquad (41) \quad (3.4)$$

In this expression the term $t_{\mu\nu}$ gives no contribution if the boundary space-time represented by the metric $g^{(0)}_{\mu\nu}$ is conformally flat. So, in cosmological applications, we can ignore this term since the FRW metric is conformally flat.

Holographic cosmology

We now seek a cosmological solution to the holographic Einstein equations such that the induced metric at the boundary has the FRW form

$$ds^2 = g^{(0)}_{\mu\nu} dx_\mu dx_\nu = dt^2 - a^2(t) d\Omega_k^2$$

From now on we assume spatial flatness, i.e., we put $\kappa = 0$. The nonvanishing components of the 3+1 dim. Ricci tensor are as usual

$$R_{00} = -3(\dot{H} + H^2) \qquad R_{ij} = a^2(\dot{H} + 3H^2)\delta_{ij}$$

And the Ricci scalar $R = -6(\dot{H} + 2H^2)$

Applying these to the 00 component of the modified Einstein equations (3.4) at the boundary we obtain the holographic Friedmann equation (Exercise No 12)

$$H^2 - \frac{\ell^2}{4} H^4 = \frac{8\pi G_\mathrm{N}}{3}\rho \qquad (42)\ (3.5)$$

quadratic
deviation

The second Friedmann equation can be derived from the continuity equation (1.9), $\dot{\rho} + 3H(\rho + p) = 0$, combined with (3.5) (Exercise No 13)

$$\dot{H}\left(1 - \frac{\ell^2}{2} H^2\right) = -4\pi G_\mathrm{N}(p + \rho) \qquad (43)\ (3.6)$$

quadratic
deviation

where $\rho = T_{00}, \quad p = -T_i^i$

E. Kiritsis, JCAP **0510** (2005); Apostolopoulos et al, Phys. Rev. Lett. **102**, (2009);
N.B., Phys. Rev. D 93 (2016), arXiv:1511.07323

The holographic cosmology has interesting properties. Solving the first Friedmann equation as a quadratic equation for H^2 we find

$$H^2 = \frac{2}{\ell^2}\left(1 \pm \sqrt{1 - 8\pi\ell^2 G_{\mathrm{N}}\rho/3}\right) \tag{3.7}$$

Demanding that this equation reduces to the standard Friedmann equation in the low energy limit, i.e., in the limit when

$$\ell^2 G_{\mathrm{N}}\rho \ll 1$$

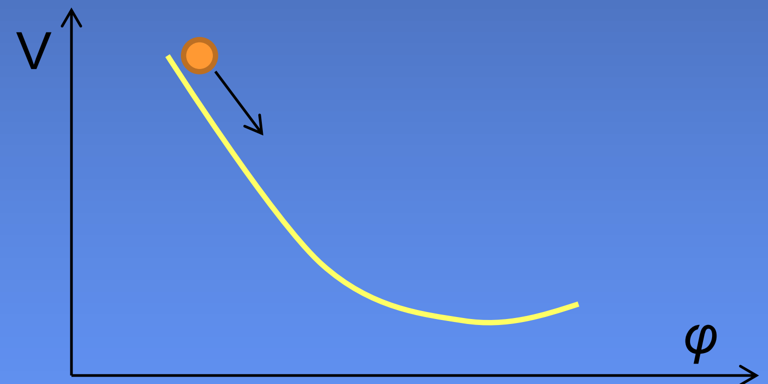
it follows that we must discard the + sign solution. Then, it follows that the physical range of the Hubble rate is between 0 and $\sqrt{2}/\ell$ starting from its maximal value $H_{\max} = \sqrt{2}/\ell$ at an arbitrary initial time t_0. At that time, which may be chosen to be zero, the density and cosmological scale are both finite so the Big-Bang singularity is avoided!

We will sometimes use the dimensionless Hubble rate

$$h = \ell H$$

Inflation

One postulates a field, dubbed the *inflaton*, usually a self-interacting scalar which evolves towards the minimum of a slow roll potential. In conjunction with Friedmann equation one solves the field equations from the beginning to the end of inflation. During inflation a **slow roll** regime is assumed, i.e., a very slow change of the Hubble rate so the Universe expands almost as a de Sitter spacetime with a large cosmological constant.



Quantum fluctuations of the inflaton field generate initial density perturbations of order $\delta\rho/\rho = 10^{-5}$ at the time of decoupling

$$t \approx 300\,000 \text{ years} \quad (z \approx 1000)$$

The main problems of the standard cosmology solved by inflation

- Horizon problem – homogeneity and isotropy of the CMB radiation
- Flatness problem – fine tuning of the initial conditions
- Large scale structure problem – the origin of initial density perturbations that serve as seeds of the observed structure today
- Monopole problem – absence of topological defects: monopoles, cosmic strings, domain walls

Inflation

During inflation the Universe evolution can be viewed as two interlinked processes: 1) Backround evolution and 2) Evolution of the cosmological perturbations

1) Backround evolution assumes isotropic and homogenous spacetime with a spatially flat FRW metric and time evolution described by the Friedmann equations as in the standard cosmology
2) Cosmological perturbations are small disturbances of the metric on top of the background. These disturbances, classified as scalar and tensor perturbations, are initiated by quantum fluctuations of the inflaton. The perturbations are calculated at linear order and quantized according to canonical quantum field theory. The spectrum of metric perturbations generated during inflation is model dependent and can be confronted with the spectrum inferred from the CMB

Background

Now we assume that the background in the holographic braneworld is a spatially flat FRW universe with line element

$$ds^2 = g^{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

and we employ the holographic Friedmann equations

$$H^2 - \frac{\ell^2}{4} H^4 = \frac{8\pi G}{3}\rho \qquad \dot{H}\left(1 - \frac{\ell^2}{2} H^2\right) = 4\pi G(p+\rho) \tag{3.8}$$

We also demand that these equations reduce to the standard Friedmann equation in the low energy limit. These equations will be solved in conjunction with the field equation for the inflaton field. We will assume the inflaton Lagrangian as a general k-essence field Lagrangian $\mathcal{L} = \mathcal{L}(X, \theta)$ with the corresponding Hamiltonian $\mathcal{H}$. The pressure and energy density are identified with the Lagrangian and Hamiltonian

$$p \equiv \mathcal{L}, \qquad \rho \equiv \mathcal{H} = 2X\mathcal{L}_X - \mathcal{L} \tag{3.9}$$

The field equation may be presented in the form of either the Euler-Lagrange equation or the Hamilton equations, as discussed in Lecture I

The most important quantities that characterize inflation are the slow-roll inflation parameters ε_j defined recursively

$$\varepsilon_{j+1} \equiv \dot{\varepsilon}_j / (H \varepsilon_j)$$

starting from $\varepsilon_0 \equiv H_* / H$, where H_* is the Hubble rate at some chosen time. The slow roll regime is characterized by $\varepsilon_i \ll 1$. The first two slow roll parameters are given by

$$\varepsilon_1 \equiv -\frac{\dot{H}}{H^2} \qquad \varepsilon_2 \equiv \frac{\dot{\varepsilon}_1}{H \varepsilon_1}$$

During inflation $\varepsilon_{1,2} < 1$ and inflation ends once either ε_1 or ε_2 exceeds 1.

Another important quantity is the so called number of e-folds defined as

$$N \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} H dt$$

where the subscripts **i** and **f** denote the beginning and the end of inflation. Typically $N \simeq 50 - 60$ is sufficient to solve the flatness and horizon problems

Cosmological perturbations

To confront our model with observation we need to calculate the power spectra of scalar and tensor cosmological perturbations $\mathcal{P}_{\mathrm{S}}$, and $\mathcal{P}_{\mathrm{T}}$, respectively, evaluated at the horizon, i.e., for a wave-number satisfying *q=aH*. We follow the formalism of J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, (1999) adjusted to account for the modfied Friedmann equations. The perturbations are calculated at linear order and quantized according to canonical quantum field theory.

Perturbations of the metric

Assuming a spatially flat background one can choose a gauge so that

$$ds^2 = (1+2\Psi)dt^2 - a^2(t)\left[(1-2\Phi)\delta_{ij} + h_{ij}\right]dx^i dx^j$$

Scalar perturbations

Tensor perturbations

The symmetric tensor h_{ij} corresponds to primordial gravitational waves generated during inflation.

Scalar perturbations

Cosider first the scalar perturbations only. Then, the perturbed line element in the Newtonian gauge

$$ds^2 = (1 + 2\Psi)dt^2 - (1 - 2\Phi)a^2(t)\delta_{ij}dx^i dx^j$$

Using this, we can easily calculate the perturbations of the Ricci tensor and Ricci scalar. The perturbed components of the Ricci tensor at linear order are

$$R_{00} = -3(H^2 + \dot{H}) + \frac{1}{a^2}\nabla^2\Psi + 3H(2\dot{\Phi} + \dot{\Psi}) + 3\ddot{\Phi},$$

$$R_{0i} = 2\partial_i(\dot{\Phi} + H\Psi),$$

$$R_{ij} = a^2(3H^2 + \dot{H})\delta_{ij} + \partial_i\partial_j(\Phi - \Psi) + \delta_{ij}\nabla^2\Phi$$
$$- a^2\delta_{ij}\left[2(3H^2 + \dot{H})(\Phi + \Psi) + H(6\dot{\Phi} + \dot{\Psi}) + H\ddot{\Phi}\right]$$

and the perturbed Ricci scalar

$$R = -6(2H^2 + \dot{H}) + \frac{2}{a^2}\nabla^2\Psi - \frac{4}{a^2}\nabla^2\Phi$$
$$+ 6\big[2(2H^2 + \dot{H})\Psi + H(4\dot{\Phi} + \dot{\Psi}) + \ddot{\Phi}\big]$$

Then, the relevant components of the perturbed Einstein equations at linear order are

$$\frac{2}{a^2}\left(1 - \frac{\ell^2}{2}H^2\right)\left(\nabla^2\Phi - 3a^2H(\dot{\Phi} + H\Psi)\right) = 8\pi G_{\rm N}\delta T^0_0 \qquad (3.10)$$

$$2\left(1 - \frac{\ell^2}{2}H^2\right)\partial_i(\dot{\Phi} + H\Psi) = 8\pi G_{\rm N}\delta T^0_i \qquad (3.11)$$

$$\left(1 - \frac{\ell^2}{2}(H^2 + \dot{H})\right)\partial^i\partial_j(\Phi - \Psi) = 0 \qquad \text{(for } i \neq j\text{)}$$

From the last equation, it follows that we can choose $\Phi = \Psi$ and work in the so-called longitudinal gauge with only one scalar, e.g., Φ

The perturbations of the stress tensor are induced by the fluctuations of the inflaton field $\delta\theta$ and metric perturbations Φ. Following standard Newtonian gauge conventions, the coordinates are chosen such that u^i is a first-order perturbative quantity ($u^i = \mathcal{O}(\delta u^i)$). Hence, up to the first perturbative order, we find

$$\delta T_0^0 = \delta\rho \qquad\qquad \delta T_i^0 = (p+\rho)\delta u_i$$

yielding

$$\delta T_0^0 = \frac{p+\rho}{c_s^2}\left[\left(\frac{\delta\theta}{\dot\theta}\right)^{\cdot} - \Phi\right] - 3H(p+\rho)\frac{\delta\theta}{\dot\theta} \qquad (3.12)$$

$$\delta T_i^0 = (p+\rho)\left(\frac{\delta\theta}{\dot\theta}\right)_{,i} \qquad (3.13)$$

where c_s is the speed of sound (see Lecture I, eq. (14a))

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_\theta = \frac{p_X}{\rho_X} = \frac{p+\rho}{2X\rho_X} \qquad (3.14)$$

In the slow roll regime the sound speed deviates slightly from unity and may be expressed in terms of slow-roll parameters. First, by making use of the definition $\varepsilon_1 = -\dot{H}/H^2$ and the modified Friedmann equations (3.8) with (3.9), we can express the variable X in the slow roll regime as

$$X = -\frac{2p(2-h^2)}{3p_X(4-h^2)}\varepsilon_1 + \mathcal{O}(\varepsilon_i^2)$$

where we introduce the dimensionless Hubble rate $h = \ell H$. Then from (3.14) we find

$$c_{\mathrm{s}}^2 = 1 + \frac{4(2-h^2)}{3(4-h^2)}\frac{p\,p_{XX}}{p_X^2}\varepsilon_1 + \mathcal{O}(\varepsilon_i^2) \qquad (3.15)$$

The standard k-essence expression will be recovered if we set $h = 0$ in (3.15)

Using (3.12) and (3.13), the modified Einstein equations (3.10) and (3.11) can be written in the form

$$\left(\frac{\delta\theta}{\dot\theta}\right)^{\cdot} = \Phi + \frac{c_{\mathrm{s}}^2}{4\pi G_{\mathrm{N}} a^2(\tilde p + \tilde\rho)}\nabla^2\Phi \qquad (3.16)$$

$$(a\Phi)^{\cdot} = 4\pi G_{\mathrm{N}} a(\tilde p + \tilde\rho)\frac{\delta\theta}{\dot\theta} \qquad (3.17)$$

Here we have defined

$$\tilde p + \tilde\rho = (p+\rho)(1-h^2/2)^{-1}$$

where $h = \ell H$.

Now we introduce new functions

$$\xi = \frac{a\Phi}{4\pi G_{\rm N} H}, \quad \zeta = \Phi + H\frac{\delta\theta}{\dot{\theta}}$$

The gauge invariant quantity ζ represents spatial curvature perturbations on uniform density (constant-θ) hypersurfaces. Substituting this into (3.16) and (3.17) and using the 2nd Friedmann eq. (3.8) we find

$$a\dot{\xi} = z^2 c_{\rm s}^2 \zeta \qquad (3.18)$$

$$a\dot{\zeta} = z^{-2}\nabla^2\xi \qquad (3.19)$$

where

$$z = \frac{a(\tilde{p}+\tilde{\rho})^{1/2}}{c_{\rm s}H} = \frac{a}{c_{\rm s}}\sqrt{\frac{\varepsilon_1}{4\pi G_{\rm N}}}$$

Here we use the definition of the slow roll parameter $\varepsilon_1 \equiv -\frac{\dot{H}}{H^2}$

By introducing a new variable $v = z\zeta$, it is straightforward to show from equations (3.18) and (3.19) that v satisfies a second order differential equation

$$v'' - c_s^2 \nabla^2 v - \frac{z''}{z} v = 0 \qquad (3.20)$$

where the primes denote derivatives with respect to the conformal time $\tau = \int dt/a$. In the slow-roll regime one can use the relation

$$\tau = -\frac{1+\varepsilon_1}{aH} + \mathcal{O}(\varepsilon_1^2)$$

which follows from the definition of ε_1 expressed in terms of the conformal time τ . At linear order in ε_i we find

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\tau^2} \qquad \text{where} \qquad \nu^2 = \frac{9}{4} + 3\varepsilon_1 + \frac{3}{2}\varepsilon_2$$

By making use of the Fourier transformation

$$v(\tau, \boldsymbol{x}) = \frac{1}{(2\pi)^3} \int d^3 q e^{i \boldsymbol{q} \boldsymbol{x}} v_q(\tau)$$

where q is a comoving wavenumber, we obtain the mode-function equation

$$v_q'' + \left(c_s^2 q^2 - \frac{z''}{z} \right) v_q = 0 \qquad (3.21)$$

There is a characteristic scale given by the acoustic horizon size related to the Hubble scale during inflation, c_s/H. There will be modes v_q with physical wavelengths a/q much smaller than this scale, or $c_s q \gg aH$, and modes with wavelengths much larger than the acoustic horizon, or $c_s q \ll aH$.
In these two asymptotic regimes, the solutions can be written as

$$v_q = \frac{1}{\sqrt{2 c_s q}} e^{-i c_s q \tau} \quad \text{for} \quad c_s q \gg aH$$

$$v_q = C z \quad \text{for} \quad c_s q \ll aH \qquad (3.22)$$

Since $v = z\zeta$, the perturbation modes ζ_q are frozen for wavelength larger than the acoustic horizon scale. The crossover regime $q \cong aH$ is called the **horizon crossing**.

The solution to (3.21) which for large τ agrees with the asymptotic functions (3.22) is

$$v_q = \frac{\sqrt{\pi}}{2}(-\tau)^{1/2} H_\nu^{(1)}(-c_{\rm s} q\tau)$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind of rank ν. In the limit of the de Sitter background $H = \text{const}$, all ε_i vanish so $\nu = 3/2$, in which case the solution is

$$v_q = \frac{e^{-ic_{\rm s}q\tau}}{\sqrt{2c_{\rm s}q}}\left(1 - \frac{i}{c_{\rm s}q\tau}\right)$$

Quantum origin of perturbations

As we have mentioned earlier, initial density perturbations are induced by quantum fluctuations of the inflaton field θ. To quantize the perturbations Φ and $\delta\theta$ we start from the action

$$S[v] = \frac{1}{2} \int d\tau d^3x \left(v'^{\,2} - c_{\mathrm{S}}^2 (\nabla v)^2 + \frac{z''}{z} v^2 \right)$$

For the scalar field v. The variation of this action obviously yields Eq. (3.20) as the equation of motion for v. Applying the standard canonical quantization the field v_q is promoted to an operator

$$\hat{v}_q = v_q \hat{a}_{\boldsymbol{q}} + v^*_{-q} \hat{a}^\dagger_{-\boldsymbol{q}}$$

where the operators $\hat{a}_q$ and $\hat{a}_q^\dagger$ satisfy the canonical commutation relations

$$[\hat{a}_{\boldsymbol{q}}, \hat{a}^\dagger_{\boldsymbol{q}'}] = (2\pi)^3 \delta(\boldsymbol{q} - \boldsymbol{q}')$$

Then, the power spectrum of the field $\zeta_q = v_q/z$ is obtained from the two-point correlation function

$$\langle \hat{\zeta}_q \hat{\zeta}_{q'} \rangle = \langle \hat{v}_q \hat{v}_{q'} \rangle / z^2 = (2\pi)^3 \delta(\boldsymbol{q} + \boldsymbol{q}') |\zeta_q|^2$$

The dimensionless spectral density

$$\mathcal{P}_{\mathrm{S}}(q) = \frac{q^3}{2\pi^2} |\zeta_q|^2 = \frac{q^3}{2\pi^2 z^2} |v_q|^2$$

characterizes the primordial scalar fluctuations, precisely as in the standard k-essence inflation. The difference with respect to the standard expression is in the modified dependence on ε_1 of the speed of sound (3.15) that appears in the definition of z

Tensor perturbations

are related to the production of gravitational waves during inflation.
The metric perturbation are defined as

$$ds^2 = dt^2 - a^2(t)\left(\delta_{ij} + h_{ij}\right) dx^i dx^j$$

Owing to the diffeomorphism invariance, the symmetric tensor h_{ij} can be made traceless, i.e., $h_i^i = 0$, and transverse, i.e., $\nabla^i h_{ij} = 0$.

In the absence of anisotropic stress, the gravitational waves are decoupled from matter and the relevant Einstein equations at linear order become

$$h''_{ij} + 2aHh'_{ij} - \Delta h_{ij} = 0$$

To solve this one uses the standard Fourier decomposition

$$h_{ij}(\tau, \boldsymbol{x}) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\boldsymbol{q}\boldsymbol{x}} \sum_s h_q^s(\tau) e_{ij}^s(q)$$

where the polarization tensor $e_{ij}^s(q)$ satisfies $q^i e_{ij}^s(q) = 0$, and $e_{ij}^s(q)\; e_{ij}^t(q) = 2\delta_{st}$, with two polarizations $s = +, \times$.

Tensor perturbations

We now introduce a canonically normalized variable

$$v_q = \frac{a}{16\pi G_{\mathrm{N}}} h_q$$

where the dependence on *s* is suppressed but we have to bear in mind to sum over two polarizations in the final expression. We obtain the mode equation

$$v_q{}'' + \left(q^2 - \frac{a''}{a}\right) v_q = 0$$

Then, the properly normalized solution can be expressed in terms of the Hankel functions

$$v_q = \frac{\sqrt{\pi}}{2}(-\tau)^{1/2} H_\nu^{(1)}(-q\tau)$$

where $\nu^2 = 9/4 + 3\varepsilon_1$

The quantization proceeds in a similar way as in the scalar case and the power spectrum of the field obtained from the two-point correlation function

$$\langle \hat{h}_q \hat{h}_{q'} \rangle = \langle \hat{v}_q \hat{v}_{q'} \rangle \frac{(16\pi G_{\mathrm{N}})^2}{a^2} = (2\pi)^3 \delta(\boldsymbol{q} + \boldsymbol{q}') |h_q|^2$$

The dimensionless spectral density which characterizes the primordial tensor fluctuations is then given by

$$\mathcal{P}_{\mathrm{T}}(q) = \frac{q^3}{\pi^2} |h_q|^2 = \frac{q^3}{\pi^2} \left| \frac{16\pi G_{\mathrm{N}}}{a} v_q \right|^2$$

with no deviation from the standard expression

Next, we evaluate the spectral densities at the horizon crossing, i.e., for a wave-number satisfying $q = aH$. Following the standard procedure we make use of the expansion of the Hankel function for $|c_s q\tau| \ll 1$

$$H_\nu^{(1)}(-c_{\mathrm{s}}q\tau) \simeq -\frac{i}{\pi}\Gamma(\nu)\left(\frac{-c_{\mathrm{s}}q\tau}{2}\right)^{-\nu}$$

where the conformal time $\tau < 0$ and q is the comoving wave number. At the lowest order in ε_1 and ε_2 we find

$$\mathcal{P}_{\mathrm{S}} \simeq \frac{G_{\mathrm{N}}H^2}{\pi c_{\mathrm{s}}\varepsilon_1}\left[1 - 2\left(1 + C\right)\varepsilon_1 - C\varepsilon_2\right]$$

$$\mathcal{P}_{\mathrm{T}} \simeq \frac{16 G_{\mathrm{N}}H^2}{\pi}\left[1 - 2\left(1 + C\right)\varepsilon_1\right]$$

where $C = \gamma - 2 + \ln 2 = -0.73$ and γ is the Euler constant, so we recover the standard expressions. However, it should be stressed again that the relation between the sound speed and slow roll parameters ε_i deviates from the standard relation, as shown in Eq. (3.15).

Scalar spectral index and tensor to scalar ratio

To confront a particular inflation model with CMB observations it is convenient to use the scalar spectral index and tensor to scalar ratio defined as

$$n_{\mathrm{S}} - 1 = \frac{d\ln \mathcal{P}_{\mathrm{S}}}{d\ln q} \simeq \frac{1}{H(1-\varepsilon_1)}\frac{d\ln \mathcal{P}_{\mathrm{S}}}{dt} \qquad\qquad r = \frac{\mathcal{P}_{\mathrm{T}}}{\mathcal{P}_{\mathrm{S}}}$$

where $\mathcal{P}_S$ and $\mathcal{P}_T$ are evaluated at the horizon crossing with $q = aH$

Keeping the terms up to the quadratic order in ε_i we obtain

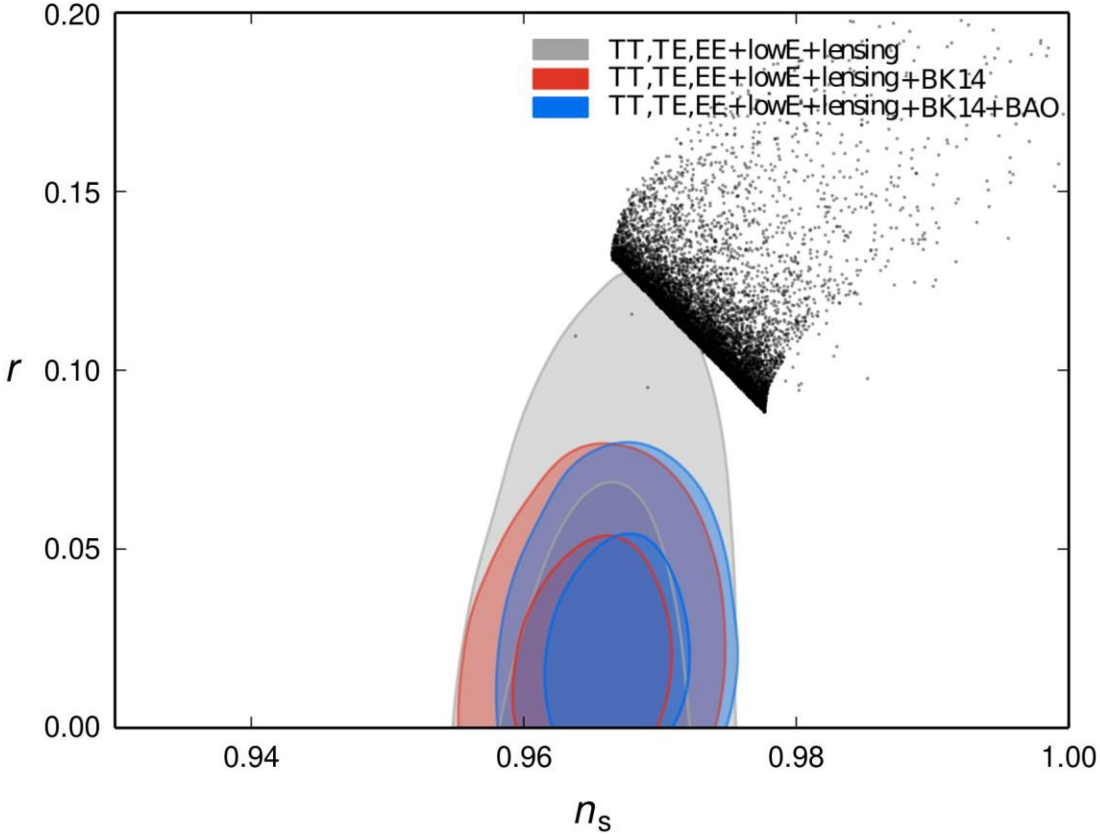
$$r = 16\varepsilon_1\left[1 + C\varepsilon_2 + \frac{2(2-h^2)}{3(4-h^2)}\frac{pp_{XX}}{p_X^2}\varepsilon_1\right]$$

$$n_{\mathrm{s}} = 1 - 2\varepsilon_1 - \varepsilon_2 - \left(2 + \frac{8h^2}{3(4-h^2)^2}\frac{pp_{XX}}{p_X^2}\right)\varepsilon_1^2 - \left(3 + 2C + \frac{2(2-h^2)}{3(4-h^2)}\frac{pp_{XX}}{p_X^2}\right)\varepsilon_1\varepsilon_2 - C\varepsilon_2\varepsilon_3$$

Example: tachyon as an inflaton

$$\mathcal{L} = -\ell^{-4} e^{-\omega\theta/\ell} \sqrt{1 - X}$$

$$0 < \omega < 0.25$$
$$60 < N < 90$$
$$0 < h_{\rm in}^2 < 2$$



Plot obtained courtesy of Milan Milosevic

Thank you

Example: tachyon as an inflaton

The existence of tachyons in the perturbative spectrum of string theory, both open and closed, indicates that the perturbative vacuum is unstable and that there exists a true vacuum towards which a tachyon field tends. The basics of this process is captured by an effective field theory model with a Lagrangian of the Dirac-Born-Infeld (DBI) form

$$\mathcal{L} = -\ell^{-4} V(\theta/\ell)\sqrt{1-X} \qquad \text{where} \qquad X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

and ℓ is an arbitrary length scale introduced to make the potential V dimensionless . The potential V is a positive function of θ with a unique local maximum at $\theta = 0$ and a global minimum at $\theta = \infty$ at which V vanishes.

A. Sen, JHEP 9910, 008 (1999) [hep-th/9909062].

Modified Gauss-Bonnet gravity

Holographic cosmology appears also in other contexts. In particular it can be derived in a modified gravity theory of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_{\mathrm{N}}}(-R + f(R,\mathcal{G})) + \mathcal{L}^{\mathrm{matt}}\right]$$

where $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss-Bonnet invariant

If in addition one requires that the second Friedmann equation is linear in $\dot{H}$, - the requirement which cannot be fulfilled in a simple $f(R)$ modified gravity including the Starobinski model - then f can be expressed as a function of only one variable $f = f(J)$ where

$$J = \frac{1}{\sqrt{12}}\left(-R + \sqrt{R^2 - 6\mathcal{G}}\right)^{1/2}$$

C. Gao, Phys. Rev. D 86 (2012)

In a cosmological context with spatially flat metric one finds $J = \dot{a}/a \equiv H$, the function f becomes a function of H, and the first Friedmann equation takes the form

$$H^2 + \frac{1}{6} f(H) - \frac{1}{6} H \frac{df}{dH} = \frac{8\pi G_{\mathrm{N}}}{3} \rho$$

The left-hand side is a function of H only and takes the holographic form if

$$f(H) = \frac{1}{2} \ell^2 H^4$$

Hence, the holographic cosmology is reproduced in the modified Gauss-Bonnet gravity of the form

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_{\mathrm{N}}} \left(-R - \frac{\ell^2}{288} \left(\sqrt{R^2 - 6\mathcal{G}} - R \right)^2 \right) + \mathcal{L}^{\mathrm{matt}} \right]$$

The tachyon model belongs to a general class of models called *k-essence* with noncanonical dependence on the kinetic term. Tachyon model and other k-essence type of models have been widely exploited as model for dark energy, unified DE/DM models, and models for inflation.

Our model is based on a holographic braneworld scenario with an effective tachyon field on a D3-brane located at the holographic boundary of ADS_5. In this model we naturally identify ℓ as the curvature radius of AdS_5 .

As we have discussed previously in Lecture I, the covariant Hamiltonian corresponding to $\mathcal{L}$ is

$$\mathcal{H} = \sqrt{\ell^{-8}V^2 + \pi_\theta^2}$$

where π_θ is the variable conjugate to $\sqrt{X}$, i.e.,

$$\pi_\theta = \frac{\partial \mathcal{L}}{\partial \sqrt{X}}$$

Tachyon inflation is based upon the slow evolution of the field θ
The slow-roll conditions $\varepsilon_{1,2} \ll 1$ are met if

$$\dot{\theta}^2 \ll 1, \qquad |\ddot{\theta}| \ll 3H\dot{\theta}.$$

Then, during the slow-roll regime we find

$$h^2 \equiv H^2\ell^2 \simeq 2(1 - \sqrt{1 - \kappa^2 V/3}),$$

where $\kappa^2 = 8\pi G / \ell^2$

and the evolution is constraint to the physical range of the Hubble rate

$$0 \le h^2 \le 2$$

In the following we will examine a simple exponential potential

$$V = e^{-\omega\theta/\ell}$$

where ω is a free dimensionless parameter. We will also consider the initial value h_i^2 as a free parameter ranging between 0 and 2.

The slow-roll parameters can be analytically calculated in the slow roll approximation. The firs two are given by

$$\varepsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{\omega^2(4-h^2)}{12h^2(2-h^2)}, \quad \varepsilon_2 \equiv \frac{\dot{\varepsilon}_1}{H\varepsilon_1} = 2\varepsilon_1\left(1-\frac{2h^2}{(2-h^2)(4-h^2)}\right)$$

$$\varepsilon_{1,2} < 1$$

During inflation $\varepsilon_{1,2} < 1$ and inflation ends once either ε_1 or ε_2 exceeds 1. Near the end of inflation $h^2 \simeq \kappa^2 V/3 \ll 1$ and $\varepsilon_2 \simeq 2\varepsilon_1$.

Another important quantity is the so called number of e-folds defined as

$$N \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} H dt$$

where the subscripts **i** and **f** denote the beginning and the end of inflation. Typically $N \simeq 50-60$ is sufficient to solve the flatness and horizon problems

From the field equations we find an approximate equation

$$\dot{\theta} \simeq \frac{\omega}{3h} \,, \qquad \text{where} \quad h \equiv H\ell$$

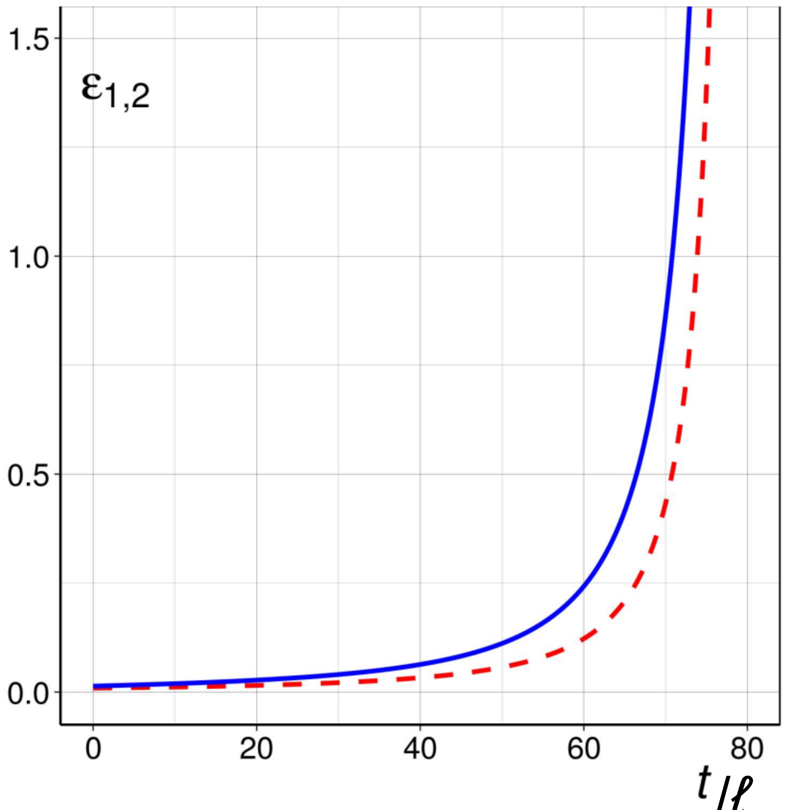
which can be easily integrated yielding the time as a function of *H* in the slow roll regime

$$t = \frac{3}{\omega^2}\left[2(h_i - h) + \ln\frac{(2-h_i)(2+h)}{(2+h_i)(2-h)}\right]$$

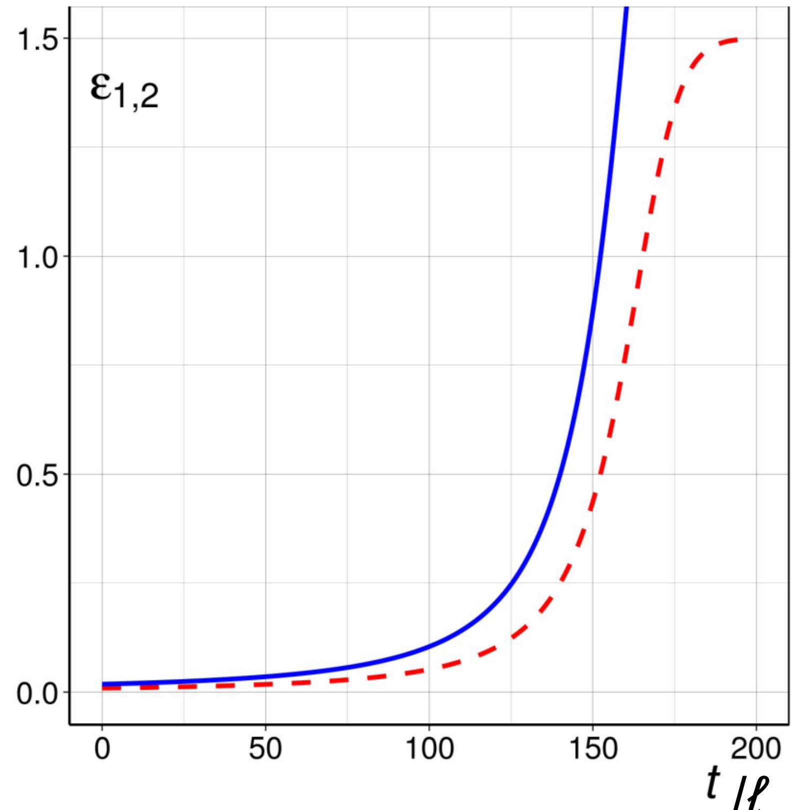
The number of e-folds can also be calculated explicitly yielding an expression that relates our free parameters h_i and ω to N

$$N = \frac{12}{\omega^2}\left[\sqrt{1-\frac{\omega^2}{3}} - 1 + \frac{h_i^2}{2} + \ln\left(2 - \frac{h_i^2}{2}\right) - \ln\left(1 + \sqrt{1-\frac{\omega^2}{3}}\right)\right]$$

Hence, for a fixed chosen *N* we have only one free parameter



Slow roll analytical



Exact numerical

Slow roll parameters ε_1 (dashed red line) and ε_2 (blue line) versus time for fixed N=60 and ω^2=0.027 corresponding to the initial h_i^2=0.6

Scalar perturbations

For scalar perturbations we introduce the perturbed line element in the longitudinal gauge

$$ds^2 = (1 + 2\Psi)dt^2 - (1 - 2\Phi)a^2(t)(dr^2 + r^2 d\Omega^2)$$

Then, the Einstein equations at linear order take the form

$$\dot{\xi} = a\frac{p+\rho}{H^2}\zeta - \frac{h\dot{h}}{2}\xi,$$

New terms

$$\dot{\zeta} = \frac{c_{\rm s}^2 H^2}{a^3(p+\rho)}\Delta\xi + \frac{\dot{h}}{2}\left(\zeta - \frac{4\pi G_{\rm N}}{a}H\xi\right)$$

where we have introduced functions

$$\xi = \frac{a\Phi}{4\pi G H}, \quad \zeta = \Phi + H\frac{\delta\theta}{\dot{\theta}}$$

The quantity ζ is gauge invariant and measures the spatial curvature of comoving (or constant-θ) hyper-surfaces.

In momentum space the relevant solution is expressed as

$$v_q = \frac{\sqrt{\pi}}{2}(-\tau)^{1/2} H_\nu^{(1)}(-c_{\rm s} q\tau)$$

τ is the conformal time and $H_\nu^{(1)}$ is the Hankel function of the first kind with

$$\nu^2 = \frac{9}{4} + \frac{3}{2}\left(2 + \frac{h^2}{2-h^2}\right)\varepsilon_1 + \frac{3}{2}\varepsilon_2$$

v_q is related to ζ as $v_q = z\zeta$ with

$$z = \frac{a(p+\rho)^{1/2}}{c_{\rm s} H} = \frac{a}{c_{\rm s}}\sqrt{\frac{\epsilon_1}{4\pi G_{\rm N}}\left(1 - \frac{h^2}{2}\right)}$$

Applying the standard canonical quantization we obtain the spectral density of the primordial scalar fluctuations

$$\mathcal{P}_{\rm S}(q) = \frac{q^3}{2\pi^2}|\zeta_q|^2 = \frac{q^3}{2\pi^2 z^2}|v_q|^2$$

.

We find at the lowest order in ε_1 , and ε_2

$$\mathcal{P}_{\mathrm{S}} = \frac{\kappa^2 h^2}{4\pi^2 (2-h^2) c_{\mathrm{s}} \varepsilon_1} \left[1 - 2\left(1 + C + \frac{C h^2}{2-h^2} \right) \varepsilon_1 - C\varepsilon_2 \right]$$

$$\mathcal{P}_{\mathrm{T}} = \frac{2\kappa^2 h^2}{\pi^2} \left[1 - 2\left(1 + C \right) \varepsilon_1 \right]$$

Deviations from the standard tachyon inflation

$$C = -2 + \ln 2 + \gamma \approx -0.72$$

$$c_{\mathrm{s}}^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_s = 1 - \frac{8-4h^2}{12-3h^2}$$

We now calculate the observational quantities such as the tensor-to-scalar ratio r and the scalar spectral index n_s defined by

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_S}, \qquad n_s = \frac{d \ln \mathcal{P}_S}{d \ln q}$$

where $\mathcal{P}_S$ and $\mathcal{P}_T$ are evaluated at the horizon, i.e., for a wave-number satisfying $q=aH$. Keeping the terms up to the 2nd order in ε_1, ε_2, and ε_3, one finds

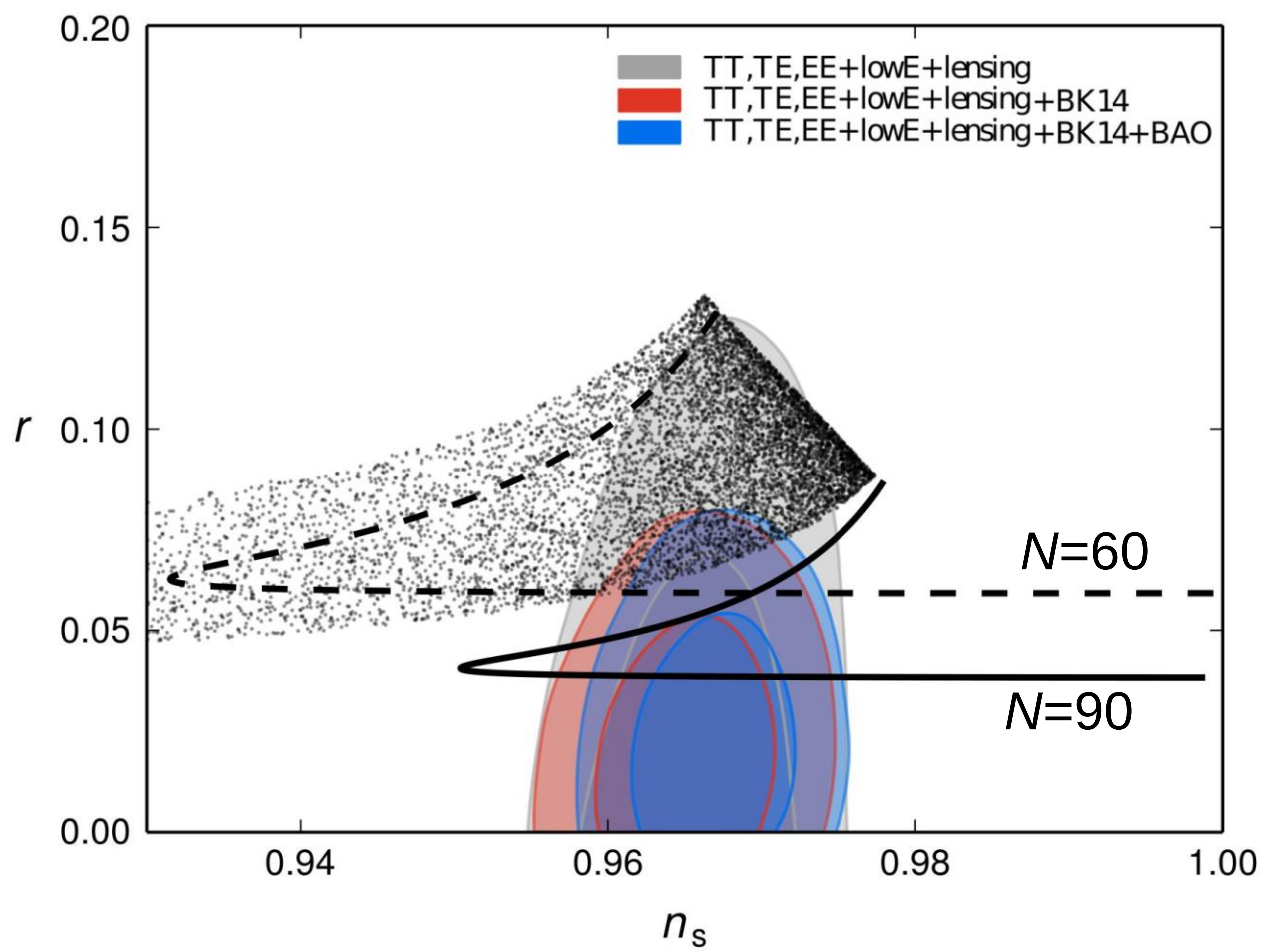
$$r = 16(1-h^2)\varepsilon_1\left(1 + C\varepsilon_2 - \frac{4-2h^2}{12-3h^2}\varepsilon_1\right)$$

Deviations from the standard tachyon inflation

$$n_s - 1 = -\left(2 + \frac{2h^2}{2-h^2}\right)\varepsilon_1 - \varepsilon_2 - \left(2 + \frac{2h^2}{2-h^2} - \frac{8h^2}{3\left(4-h^2\right)^2}\right)\varepsilon_1^2$$

$$-\frac{1}{3}\left(8 + 6C + \frac{h^2}{4-h^2}\right)\varepsilon_1\varepsilon_2 - C\varepsilon_2\varepsilon_3$$

This can be confronted with Planck 2018 observations



r versus n_s with observational constraints are from Planck 2018. The dots represent theoretical predictions obtained numerically for randomly chosen N ranging between 60 and 90 and h_i^2 between 0 to 2. The parameter ω is also varying in view of the functional dependence $N=N(h_i, \omega)$. The black lines represent the analytical results in the slow roll approximation.

Conclusions and outlook

- The slow-roll equations of the tachyon inflation with exponentially attenuating potential on the holographic brane show substantial deviations from those of the standard tachyon inflation with the same potential
- The n_s - r relation depends on the initial value of the Hubble rate and on the assumed value of the number of e-folds N and show a reasonable agreement with the Planck 2018 data for $N > 60$.
- The presented results are encouraging. What remains to be done is to solve the exact equations numerically for various other potentials that have been exploited in the literature.
- We have carried out the analysis of non-Gaussianity. Our estimate shows no essential deviations with respect to standard tachyon models

Thank you

Basic idea

Braneworld cosmology is based on the scenario in which matter is confined on a brane in a higher dimensional bulk with only gravity allowed to propagate in the bulk. The brane can be placed, e.g., at the boundary of a 5-dim asymptotically Anti de Sitter space (AdS_5)

Anti de Sitter space is dual to a conformal field theory at its boundary (AdS/CFT correspondence)
AdS is a maximally symmetric solution to Einstein's equations with negative cosmological constant. In 4+1 dimensions the symmetry group is $AdS_5 \equiv SO(4,2)$

The 3+1 boundary conformal field theory is invariant under conformal transformations: Poincare + dilatations + special conformal transformation = conformal group $\equiv SO(4,2)$

Consider a 5-dim bulk action in asymptotic AdS$_5$ background

$$S_{(5)}[\Phi] = \int d^5x \sqrt{-G}\,\mathcal{L}_{(5)}(\Phi, G_{ab})$$

Given the boundary value $\varphi(x) \equiv \Phi(z=0, x)$ and the induced metric on the boundary $h_{\mu\nu}$, the bulk field Φ and the metric G_{ab} are completely determined by field equations obtained from the variational principle

$$\frac{\delta S_{(5)}}{\delta G_{ab}} = 0 \qquad \frac{\delta S_{(5)}}{\delta \Phi} = 0$$

.

Using the solution $\Phi = \Phi[\varphi, h]$, we can define a functional

$$S[\varphi, h] = S^{\text{shell}}\big[\Phi[\varphi, h]\big]$$

where $S^{\text{shell}}\big[\Phi[\varphi, h]\big]$ is the **on-shell** bulk action, i.e., the action in which the fields are solutions of the equations of motion given their boundary values. The on-shell bulk action is still subject to the variation of the boundary values.

AdS/CFT conjecture: $S[\varphi,h]$ can be identified with the generating functional of a conformal field theory on the boundary

$$S[\varphi,h] \equiv \ln \int d\psi \exp\left\{-\int d^4x\sqrt{-h}\left[\mathcal{L}^{\mathrm{CFT}}(\psi(x)) - O(\psi(x))\varphi(x)\right]\right\}$$

where the boundary fields serve as sources for CFT operators

$\mathcal{L}^{\mathrm{CFT}}(\psi)$ – conformal field theory Lagrangian

$O(\psi)$ – operators of dimension Δ

In this way the CFT correlation functions can be calculated as functional derivatives of the on-shell bulk action, e.g.,

$$\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} = \left\langle O(\psi(x))O(\psi(y))\right\rangle - \left\langle O(\psi(x))\right\rangle\left\langle O(\psi(y))\right\rangle$$

The induced metric $h_{\mu\nu}$ serves as the source for the stress tensor of the dual CFT so that its vacuum expectation value is obtained as

$$\frac{1}{2\sqrt{-h}}\frac{\delta^2 S}{\delta h^{\mu\nu}} = \left\langle T_{\mu\nu}^{\mathrm{CFT}}\right\rangle$$

The Planck mass scale is determined by the curvature of the five-dimensional space-time

$$\frac{1}{G_{\mathrm{N}}} = \frac{\gamma}{G_5}\int_0^{\infty} e^{-2y/\ell}\,dy = \frac{\gamma\ell}{2G_5} \qquad \gamma = \begin{cases} 1 & \text{one-sided} \\ 2 & \text{two-sided} \end{cases}$$

One usually imposes the RS fine tuning condition

$$\sigma = \sigma_0 \equiv \frac{3\gamma}{8\pi G_5 \ell} = \frac{3}{8\pi G_{\mathrm{N}} \ell^2}$$

which eliminates the 4-dim cosmological constant.

Bound on the AdS$_5$ curvature radius ℓ:

The classical 3+1 dim gravity is altered on the RSII brane
For $r \gg \ell$ the Newtonian potential of an isolated source on the brane is given by

$$\Phi(r) = \frac{G_{\mathrm{N}} M}{r}\left(1 + \frac{2\ell^2}{3r^2}\right)$$

J. Garriga and T. Tanaka, Phys. Rev. Lett. 84, 2778 (2000)

Table top tests of Long et al find no deviation of Newton’s potential and place the limit

$$\ell < 0.1\,\mathrm{mm} \quad \text{or} \quad \ell^{-1} > 10^{-12}\,\mathrm{GeV}$$

Holographic cosmology

We now seek a cosmological solution to the holographic Einstein equations such that the induced metric at the boundary has the FRW form

$$ds^2 = g^{(0)}_{\mu\nu} dx_\mu dx_\nu = dt^2 - a^2(t) d\Omega_k^2$$

First, we represent the bulk metric in AdS-Schwarzschild static coordinates $(\tau, r, \chi, \vartheta, \varphi)$

$$ds^2_{(5)} = f(r) d\tau^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_\kappa^2$$

where

$$f(r) = \frac{r^2}{\ell^2} + \kappa - \mu \frac{\ell^2}{r^2} \qquad \mu = \frac{8G_5 M_{\mathrm{bh}}}{3\pi\ell^2}$$

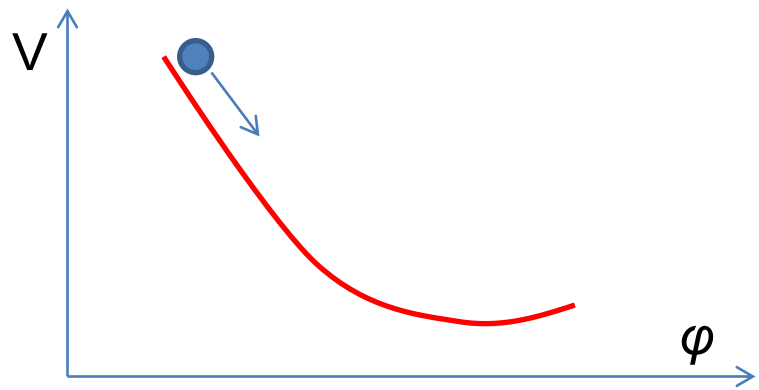
$$d\Omega_\kappa^2 = d\chi^2 + \frac{\sin^2(\sqrt{\kappa}\chi)}{\kappa}(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

Holographic type cosmologies appear also in other contexts:

1. The saddle point of the spatially closed mini superspace partition function dominated by matter fields conformally coupled to gravity
A. O. Barvinsky, C. Deffayet and A. Y. Kamenshchik, JCAP **0805**, (2008) arXiv:0801.2063
2. A modified Friedmann equation with a quartic term $\sim H^4$ derived from the generalized uncertainty principle and the first low of thermodynamics applied to the apparent horizon entropy.
J. E. Lidsey, Phys. Rev. D **88** (2013) arXiv:0911.3286
3. The quartic term as a quantum correction to the Friedmann equation using thermodynamic arguments at the apparent horizon [37]
S. Viaggiu, Mod. Phys. Lett. A, **31** (2016) arXiv:1511.06511
4. Modified Gauss-Bonnet gravity
G. Cognola et al Phys. Rev. D 73 (2006), C. Gao, Phys. Rev. D 86 (2012);

Inflation

One postulates a field, dubbed the *inflaton,* usually a self-interacting scalar that evolves towards the minimum of a slow roll potential. In conjunction with Friedman equation one solves the field equations from the beginning to the end of inflation. During inflation a slow roll regime is assumed, i.e., a very slow change of the Hubble rate so the Universe expands almost as a de Sitter spacetime with a large cosmological constant.



Quantum fluctuations of the inflaton field generate initial density perturbations of order $\delta\rho/\rho = 10^{-5}$ at the time of decoupling

$$t \approx 300\,000\,\text{years} \quad (z \approx 1000)$$

Solving Einstein's equations in the bulk one finds

$$\mathcal{A}^2 = a^2\left[1-\left(H^2+\frac{\kappa}{a^2}\right)\frac{z^2}{4}\right]^2+\frac{1}{4}\frac{\mu z^4}{a^4}, \quad \mathcal{N} = \frac{\dot{\mathcal{A}}}{\dot{a}},$$

where $H \equiv \dot{a}/a$ is the Hubble rate at the boundary and μ is the dimensionless parameter related to the bulk BH mass

P.S. Apostolopoulos, G. Siopsis, and N. Tetradis, Phys. Rev. Lett. **102**, (2009)

P. Brax and R. Peschanski, Acta Phys. Polon. **B 41** (2010)

Comparing the exact solution with the expansion

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + z^2 g_{\mu\nu}^{(2)} + z^4 g_{\mu\nu}^{(4)} + \cdots$$

we can extract $g_{\mu\nu}^{(2)}$ and $g_{\mu\nu}^{(4)}$. Then, using the de Haro et al. expression for T^{CFT} we obtain

Starting from AdS-Schwarzschild static coordinates
and making the coordinate transformation $\tau = \tau(t,z),\ r = r(t,z)$
the bulk line element will take a general form

$$ds^2_{(5)} = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2) = \frac{\ell^2}{z^2}\left[\mathcal{N}^2(t,z)dt^2 - \mathcal{A}^2(t,z)d\Omega_k^2 - dz^2\right]$$

Imposing the boundary conditions at z=0:

$$\mathcal{N}(t,0) = 1, \quad \mathcal{A}(t,0) = a(t)$$

the induced metric at the boundary takes the FRW form

$$ds^2 = dt^2 - a^2(t)d\Omega_k^2$$

$$\left\langle T^{\mathrm{CFT}}_{\mu\nu} \right\rangle = t_{\mu\nu} + \frac{1}{4}\left\langle T^{\mathrm{CFT}\,\alpha}{}_{\alpha} \right\rangle g^{(0)}_{\mu\nu}$$

The second term due to the conformal anomaly and is given by

$$\left\langle T^{\mathrm{CFT}\,\alpha}{}_{\alpha} \right\rangle = \frac{3\ell^3}{16\pi G_5}\frac{\ddot{a}}{a}\left(H^2 + \frac{\kappa}{a^2}\right)$$

The first term is a traceless tensor with non-zero components

$$t_{00} = -3t^i_i = \frac{3\ell^3}{64\pi G_5}\left[\left(H^2 + \frac{\kappa}{a^2}\right)^2 + \frac{4\mu}{a^4} - \frac{\ddot{a}}{a}\left(H^2 + \frac{\kappa}{a^2}\right)\right]$$

Hence, apart from the conformal anomaly, the CFT dual to the time dependent asymptotically AdS$_5$ metric is a conformal fluid with the equation of state $p_{\mathrm{CFT}} = \rho_{\mathrm{CFT}}/3$

where $\rho_{\mathrm{CFT}} = t_{00}$ $\quad p_{\mathrm{CFT}} = -t^i_i$

N.B., Phys. Rev. D 93 (2016), arXiv:1511.07323

Tachyon inflation

One of the popular models of inflaton is the *tachyon*. Our aim is to study tachyon inflation in the framework of holographic cosmology . The model is based on a holographic braneworld scenario with an effective tachyon field on the D3-brane located at the holographic bound of ADS bulk. A tachyon Lagrangian of the form

$$\mathcal{L} = -\ell^{-4} V(\theta / \ell)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

can be derived in the context of a dynamical brane moving in a 4+1 background with a general warp

$$ds^2_{(5)} = \frac{1}{\chi^2(z)}(g^{\mu\nu}dx^\mu dx^\nu - dz^2)$$

The field θ is identified with the 5-th coordinate z and the potential is related to the warp

$$V(\theta / \ell) = \ell^4 / \chi^4(\theta)$$

N.B., S. Domazet and G. Djordjevic, Class. Quant. Grav. **34**, (2017) arXiv:1704.01072.