Lecture 2 – Braneworld Universe

Hypersurfaces
Strings and Branes
Randal Sundrum Model
Braneworld Cosmology

Braneworld universe

Braneworld universe is based on the scenario in which matter is confined on a brane moving in the higher dimensional bulk with only gravity allowed to propagate in the bulk.

N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429** (1998)
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436** (1998)
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 (RS I), Phys. Rev. Lett. **83** (1999) 4690 (RS II)

Hypersurfaces

In a *d*-dimensional spacetime manifold (bulk), a hypersurface is a p-dimensional ($p<d$) submanifold (subspace) which can be either <u>spacelike</u>, <u>timelike</u>, or <u>null</u>.

X^a – coordinates in the bulk

$a,b=0,1,2,3,\ldots,d-1$

x^μ – coordinates on the hypersurface $\mu,\nu=1,2,3,\ldots,p$

In particular, we will consider submanifolds of dimension $p=d-1$. For example, in the braneworld scenario, the Universe is a 3+1-dimensional timelike hypersurface in a 4+1-dimensional bulk

Defining equations

A d-dimensional hypersurface Σ in a $d+1$-dimensional bulk with metric G_{ab} can be selected either by

1) putting a restriction on the coordinates

$$\Phi(X^a) = \mathrm{const}$$

so that $d\Phi=0$ along the hypersurface, or by

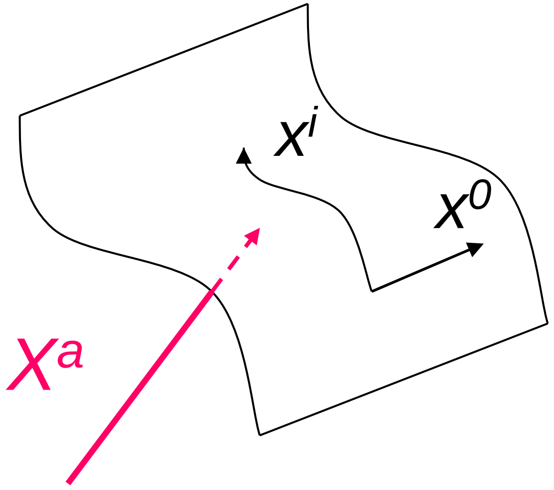
2) parametric equations of the form

$$X^a = X^a(x^\mu)$$

X^a – coordinates in the bulk

$a,b=0,1,2,3, \ldots, d$

x^μ – coordinates on the hypersurface $\mu,\nu=0,1,2,3, \ldots, d-1$



Normal vector

Because $d\Phi \equiv \Phi_{,a} dX^a = 0$, the vector $\Phi_{,a}$ is orthogonal to the displacements dX^a along Σ. Depending on the sign of the norm of the vector $\Phi_{,a}$ we have

$$G^{ab}\Phi_{,a}\Phi_{,b} \begin{cases} >0 & \text{- spacelike hypersurface} \\ <0 & \text{- timelike hypersurface} \\ =0 & \text{- null hypersurface} \end{cases}$$

If $G^{ab}\Phi_{,a}\Phi_{,b} \neq 0$, $\Phi_{,a}$ can be normalized

$$n_a = \frac{\Phi_{,a}}{\sqrt{|G^{ab}\Phi_{,a}\Phi_{,b}|}}$$

so that

$$n_a = \begin{cases} 1 & \text{for a spacelike hypersurface } (n_a \text{ is a timelike vector}) \\ -1 & \text{for a timelike hypersurface } (n_a \text{ is a spacelike vector}) \end{cases}$$

Induced metric

For the displacements dX^a on Σ we have

$$ds_\Sigma^2 = G_{ab}dX^a dX^b = G_{ab}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}dx^\mu dx^\nu = g_{\mu\nu}^{\text{ind}}\, dx^\mu dx^\nu$$

The 4-tensor

$$g_{\mu\nu}^{\text{ind}} = G_{ab}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}$$

is called the <u>induced metric,</u> or <u>first fundamental form</u>, of the hypersurface Σ. This equation can be regarded as a coordinate transformation

$$G'_{cd} = G_{ab}\frac{\partial X^a}{\partial x^c}\frac{\partial X^b}{\partial x^d}$$

from $\{X^a\}$ to $\{x^c\}$ coordinate frames and we restrict the tensor G'_{ab} to the coordinates $a,b=0,1,2,\ldots,d-1$ which we denote by Greek letters μ,ν. From now on we restrict attention to $d=4$.

Projector

The tensor

$$h_{ab} = G_{ab} - \epsilon n_a n_b \qquad a, b = 0,1,2,3,4$$

is the projection tensor onto Σ, where n_a is a unit vector normal to Σ with $\epsilon = 1$ (-1) for a timelike (spacelike) vector n_a . Clearly $n^a h_{ab} = 0$.
One can show that the 4-tensor $h_{\mu\nu}$, $\mu, \nu = 0,1,2,3,$ is also an induced metric on Σ which is related to $g_{\mu\nu}^{\mathrm{ind}}$ by a coordinate transformation.
To show this, make a coordinate transformation $X^a = X^a(\tilde{X}^b)$ such that the normal vector in the new coordinates takes the form $\tilde{n}^a = \delta^a_y$, where the coordinate y is such that the defining equation for the hypersurface Σ is $y = \mathrm{const}$. In this coordinate frame the projector onto Σ is

$$\tilde{h}_{\mu\nu} = \tilde{G}_{\mu\nu} = \tilde{g}_{\mu\nu}^{\mathrm{ind}}$$

where the bulk metric components in new coordinates are obtained as usual

$$\tilde{G}_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial \tilde{X}^\mu} \frac{\partial X^b}{\partial \tilde{X}^\nu}$$

Extrinsic curvature

The projection of the covariant derivative of n_d

$$K_{ab} = h_a^c h_b^d n_{d;c}$$

restricted to the coordinates μ,ν=0,1,2,3, i.e., the 4-tensor $K_{\mu\nu}$, is called the extrinsic curvature, or second fundamental form, of the hypersurface Σ. In special coordinates, with $n^a = \delta_y^a$, we find

$$K_{\mu\nu} = n_{\mu;\nu} = -\Gamma_{\mu\nu}^a n_a$$

The trace K of K_{ab}

$$K \equiv G^{ab} K_{ab} = h^{ab} K_{ab} = n_{;a}^a$$

Einstein-Hilbert action

In 4+1 dimensional spacetime with boundary at a 4 dim. hypersurface Σ , the vacuum Einstein equations can be derived from the action

$$S = \frac{1}{8\pi G_5} \int d^5x \sqrt{-G} \left(-\frac{R^{(5)}}{2} - \Lambda_5 \right) + S_{\mathrm{GH}}$$

The Gibbons-Hawking boundary term is

$$S_{\mathrm{GH}} = \frac{\epsilon}{8\pi G_5} \int_\Sigma d^4x \sqrt{-\det h}\,(K - K_0)$$

where $\epsilon = \pm 1$ for a timelike (spacelike) hypersurface Σ and K_0 is the trace of the extrinsic curvature of Σ embedded in flat spacetime. The GH term is necessary to cancel a generally nonvanishing contribution of the boundary in the variation of the action δS.

Then, the variation principle $\delta S=0$ yields the Einstein equations in vacuum

$$R_{ab}^{(5)} - \frac{1}{2}R^{(5)}G_{ab} = \Lambda_5 G_{ab}$$

and junction conditions

$$\left[\left[K_\beta^\alpha - K\delta_\beta^\alpha\right]\right] = 8\pi G_5 T_\beta^\alpha$$

where T_β^α is the energy momentum tensor for matter localized on the hypersurface Σ and $[[f]]$ denotes the discontinuity of a function $f(x)$ across Σ, i.e.,

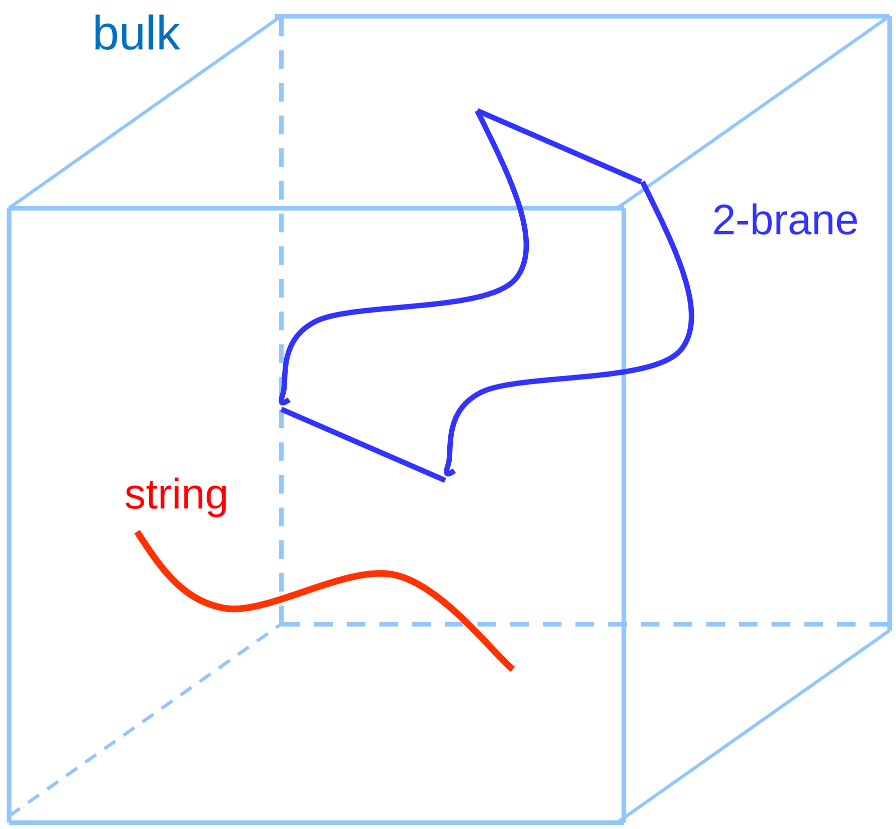
$$\left[\left[f(x)\right]\right] = \lim_{\varepsilon\to 0}(f(x+\varepsilon) - f(x-\varepsilon))$$

The junction conditions prescribe the appropriate boundary conditions across a singular hypersurface Σ supported by a localized energy momentum tensor T_β^α .

Strings and (mem)branes

STRING is a 1+1-dimdimensional object moving in the d+1 dimensional bulk

p-BRANE is a *p*+1-dim. object that generalizes the concept of membrane (2-brane) or string (1-brane)



Relativistic particle action

PARTICLE is a 0+1-dimdimensional object the dynamics of which in *d*+1-dimensional bulk is described by the relativistic pointlike-particle action

$$S_{\text{part}} = -\int \sqrt{ds^2} = -\int d\tau \sqrt{\gamma} = -\int d\tau \sqrt{1 - \dot{x}^2} \quad ,$$

where

$$ds^2 = G_{ab} dx^a dx^b, \quad \gamma = G_{ab} \frac{\partial x^a}{\partial \tau} \frac{\partial x^b}{\partial \tau}, \qquad a, b = 0, \dots, d$$

$$\dot{x}^2 = G_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} \qquad i, j = 1, \dots, d \qquad \tau \equiv x^0$$

G_{ab} – metric in the bulk
x^a – coordinates in the bulk;
τ – synchronous time coordinate ($ds^2 = d\tau^2 + G_{ij} dx^i dx^j$)

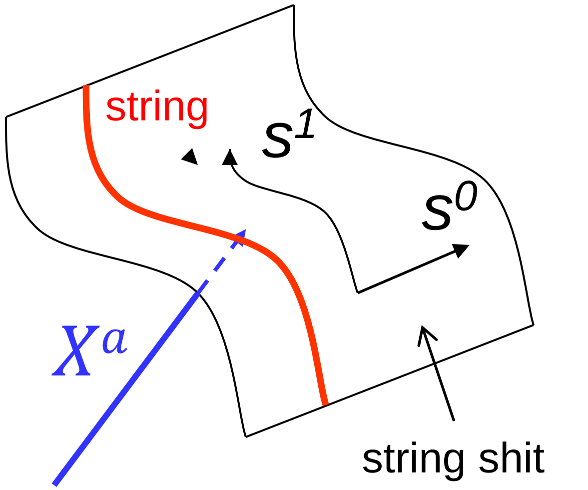
String action

The dynamics of a ***STRING*** in *d*+1-dimensional bulk is described by the Nambu-Goto action (generalization of the relativistic particle action)

$$S_{\text{string}} = -T\int d\tau d\sigma \sqrt{-\det(g_{\alpha\beta}^{\text{ind}})}$$

where $g_{\alpha\beta}^{\text{ind}}$ is the induced metric

$$g_{\alpha\beta}^{\text{ind}} = G_{ab}\frac{\partial X^a}{\partial s^\alpha}\frac{\partial X^b}{\partial s^\beta} \qquad \alpha,\beta = 0,1$$



T – string tension

X^a – coordinates in the bulk

$s^0 \equiv \tau$ – timelike coordinate on the string sheet

$s^1 \equiv \sigma$ – spacelike coordinate on the string sheet

Brane action

The dynamics of a ***p-BRANE*** in d+1-dimensional bulk is described by the Nambu-Goto action as a generalization of the string action. Nambu-Goto action for a 3-brane embedded in a 4+1-dimensional space-time (bulk)

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-\det h_{\mu\nu}}$$

$$h_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu}$$ – induced metric

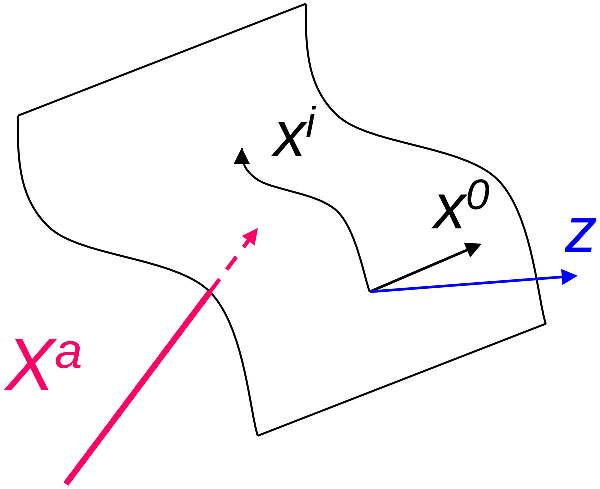
σ – brane tension

X^a – coordinates in the bulk

a,b=0,1,2,3,4

x^μ – coordinates on the brane

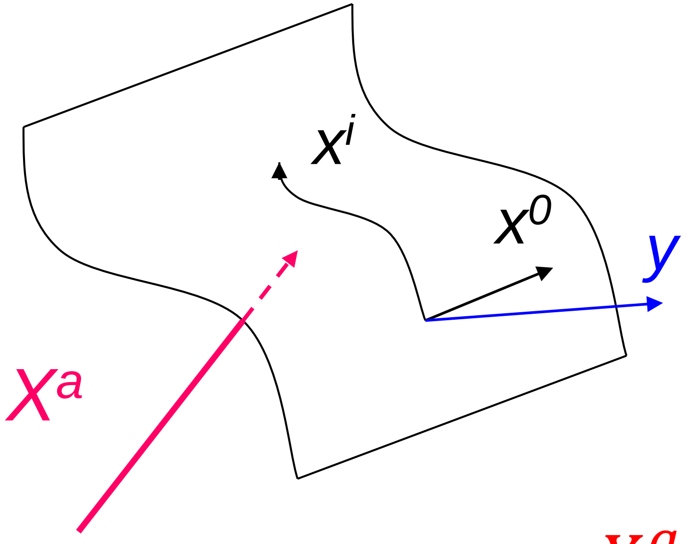
μ,ν=0,1,2,3



Example: dynamical brane as a tachyon

Consider a 3-brane moving in the 5-d bulk spacetime with metric

$$ds^2_{(5)} = \psi^2(y)\, g_{\mu\nu}dx^\mu dx^\nu - dy^2$$



$$X^a = (x^\mu, y)$$

The points on the brane can be are parameterized as $X^a = \left(x^\mu, Y(x)\right)$
The 5-th coordinate Y is treated as a dynamical field that depends on x^μ.
From the brane action

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-\det h_{\mu\nu}}$$

using the induced metric

$$h_{\mu\nu} = G_{ab} X^a_{,\mu} X^b_{,\nu} = \psi^2(Y) g_{\mu\nu} - Y_{,\mu} Y_{,\nu} \qquad \text{(a)}$$

one finds

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-g}\, \psi^4(Y) \left(1 - \psi^{-2} g^{\mu\nu} Y_{,\mu} Y_{,\nu}\right)^{1/2} \qquad \text{(b)}$$

Exercise No 7:
(i)Prove the following relation

$$\det(g_{\mu\nu} - \alpha^2 u_\mu u_\nu) = (1 - \alpha^2) \det g_{\mu\nu} \qquad \text{(c)}$$

for a general metric $g_{\mu\nu}$, unit timelike vector u_μ, and $\alpha^2 < 1$

Hint: use a comoving reference frame.

(ii) Use (c) to derive (b) from (a)

Changing Y to a new field $\theta = \int dY/\psi(Y)$ we obtain the effective brane action

$$S_{\mathrm{br}} = -\int d^4x\sqrt{-g}V(\theta)\sqrt{1-g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

Where we have defined

$$V(\theta) = \sigma\psi^4(Y(\theta))$$

This action is of the Born-Infeld type and describes the *tachyon condensate*

Exercise No 8: Show that $V(\theta) \propto \theta^{-4}$ in the AdS_5 background metric, i.e., for $\psi = e^{-y/\ell}$

Randall-Sundrum model

Randall-Sundrum proposed two scenarios with a braneworld embedded in a 5-dim asymptotically Anti deSitter space (AdS_5)

L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 (RS I)
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690 (RS II)

Anti de Sitter space AdS_5 is an empty 4+1 dimensional space-time with negative cosmological constant. The Einsten field equations in 4+1 dimensions

$$R^{(5)}_{ab} - \frac{1}{2} R^{(5)} G_{ab} = \Lambda_5 G_{ab} \qquad (2.1)$$

can be easily solved. The cosmological constant is related to the AdS_5 curvature radius ℓ

$$\Lambda_5 = -\frac{6}{\ell^2}$$

Depending on assumed symmetry of AdS$_5$ the solutions to the Einstein equations can be represented in various coordinate frames:

Gaussian normal coordinates

$$ds^2_{(5)} = G_{ab}dx^a dx^b = e^{-2y/\ell}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

Flat Fefferman-Graham coordinates $\qquad z = \ell e^{y/\ell}$

$$ds^2_{(5)} = \frac{\ell^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$$

Schwarzschild coordinates (static, spherically symmetric)

$$ds^2_{(5)} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_\kappa^2 \qquad (2.2)$$

where
$$d\Omega_\kappa^2 = d\chi^2 + \frac{\sin^2\sqrt{\kappa}\chi}{\kappa} d\Omega^2$$

$$\kappa = \begin{cases} +1 & \text{closed spherical} \\ 0 & \text{open flat} \\ -1 & \text{open hyperbolic} \end{cases}$$

$$f(r) = \frac{r^2}{\ell^2} + \kappa - \mu\frac{\ell^2}{r^2} \qquad (2.3)$$

where
$$\mu = \frac{8G_5 M_{\rm bh}}{3\pi\ell^2}$$

Hence, there is a black hole of mass $M_{\rm bh}$ at the origin.
The metric (2.2) is called the AdS Schwarzschild black hole.

RS model was proposed as an alternative to the compactification of extra dimensions. If extra dimensions were large that would yield unobserved modification of Newton's gravitational law. Experimental bound on the volume of *n* extra dimensions $V^{1/n} \leq 0.1\,\mathrm{mm}$

Long *et al*, Nature **421** (2003).

RS brane-world does not rely on compactification to localize gravity at the brane, but on the curvature of the bulk ("warped compactification"). The negative cosmological constant $\Lambda^{(5)}$ acts to "squeeze" the gravitational field closer to the brane. One can see this in Gaussian normal coordinates on the brane at $y = 0$, for which the AdS_5 metric takes the form

$$ds^2_{(5)} = G_{ab}dX^a dX^b = \underbrace{e^{-2ky}}_{\text{warp factor}}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

First Randall-Sundrum model (RS I)

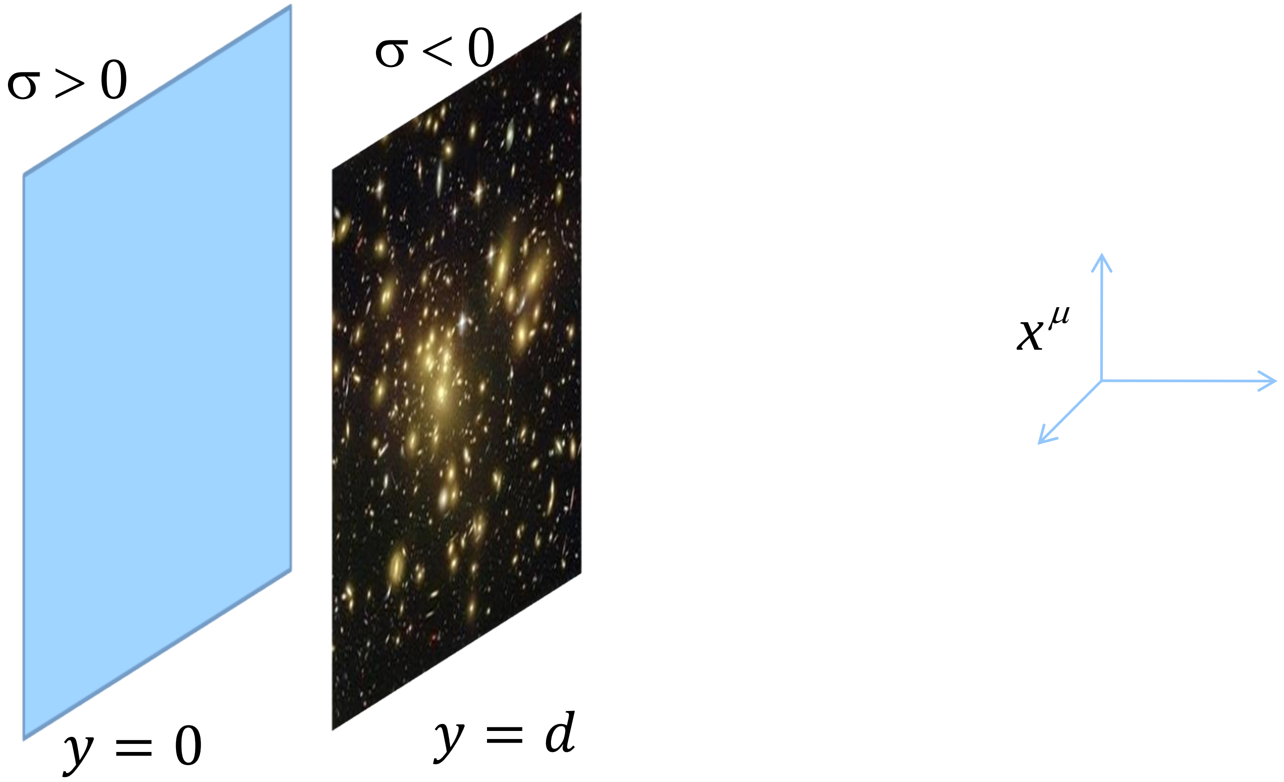
RS I was proposed as a solution to the hierarchy problem, in particular between the Planck scale $M_{Pl} \sim 10^{19}$ GeV and the electroweak scale $M_{EW} \sim 10^{3}$ GeV

RS I is a 5-dim. universe with AdS_5 geometry containing two 4-dim. branes with opposite brane tensions separated in the 5th dimension.

The observer is placed on the negative tension brane and the separation is such that the strength of gravity on observer's brane is equal to the observed 4-dim. Newtonian gravity.

L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 (RS I)

Observers reside on the negative tension brane at $y=l$



The coordinate position $y=d$ of the negative tension brane serves as a compactification radius so that the effective compactification scale is $\mu_c = 1/d$

The **conventional** approach to the hierarchy problem is to assume n compact extra dimensions with volume V_n . If their size is large enough compared to the Planck scale, i.e., if

$$\mu_{\mathrm{c}} \sim 1/V_n^{1/n} \ll M_{\mathrm{Pl}}$$

such a scenario may explain the large mass hierarchy between the **electroweak scale** M_{EW} and the **fundamental scale** M of $4+n$ gravity. In the simplest case, when the $4+n$ dim. spacetime is a product of a 4-dim. spacetime with an n-dim. compact space, one finds

$$M_{\mathrm{Pl}}^2 = M^{2+n} V_n \sim M^{2+n}/\mu_{\mathrm{c}}^n$$

In this way the fundamental $4+n$ scale M could be of the order of M_{EW} if the compactification scale satisfies

$$\mu_{\mathrm{c}}/M_{\mathrm{EW}} \sim (M_{\mathrm{EW}}/M_{\mathrm{Pl}})^{2/n}$$

Unfortunately, this introduces a new hierarchy $\mu_{\mathrm{c}} << M_{\mathrm{EW}}$

Another problem is that there exist a lower limit on the fundamental scale *M* determined by null results in table-top experiments to test for deviations from Newton's law in 4 dimensions, $U \sim 1/r$. These experiments currently probe sub-millimeter scales, so that

$$V^{1/n} \leq 0.1mm \sim \frac{1}{10^{-15}\mathrm{TeV}} \Longrightarrow M \geq \begin{cases} 5\cdot 10^{5}\mathrm{TeV} & \text{for n=1} \\ 3\mathrm{TeV} & \text{for n=2} \end{cases}$$

Long *et al*, Nature **421** (2003).

Stronger bounds for brane-worlds with compact flat extra dimensions can be derived from null results in particle accelerators and in high-energy astrophysics

M. Cavagli`a, “Black Hole and Brane Production in TeV Gravity: A Review” Int. J. Mod. Phys. A **18** (2003).

S. Hannestad and G.Raffelt, “Stringent Neutron-Star Limits on Large Extra Dimensions” Phys. Rev. Lett. **88** (2002).

In contrast to usual compactification scenarios, the RS brane-worlds do not rely on compactification of extra dimensions to localize gravity, but on the curvature of the bulk ("warped compactification"). What prevents gravity from 'leaking' into the extra dimension at low energies is a negative bulk cosmological constant

$$\Lambda_5 = -\frac{6}{\ell^2} = -6k^2$$

ℓ - curvature radius of AdS_5 corresponding to the scale $k=1/\ell$

Λ_5 acts to "squeeze" the gravitational field closer to the brane. One can see this in Gaussian normal coordinates $X^a = (x^\mu, y)$ on the brane at $y = 0$, for which the AdS_5 metric takes the form

$$ds^2_{(5)} = G_{ab} dX^a dX^b = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

Warp factor

The Planck scale is related to the fundamental scale as

$$M_{\mathrm{Pl}}^2 = 2M^3 \int_0^d e^{-2ky}\, dy = \frac{M^3}{k}\left(1 - e^{-2kd}\right)$$

So that M_{Pl} depends only weakly on d in the limit of large kd. However, any mass parameter m_0 on observer's brane in the fundamental 5-dim. theory will correspond to the physical mass

$$m = e^{-kd} m_0$$

If kd is of order 10 – 15, this mechanism produces TeV physical mass scales from fundamental mass parameters not far from the Planck scale 10^{19} GeV. In this way we do not require large hierarchies among the fundamental parameters

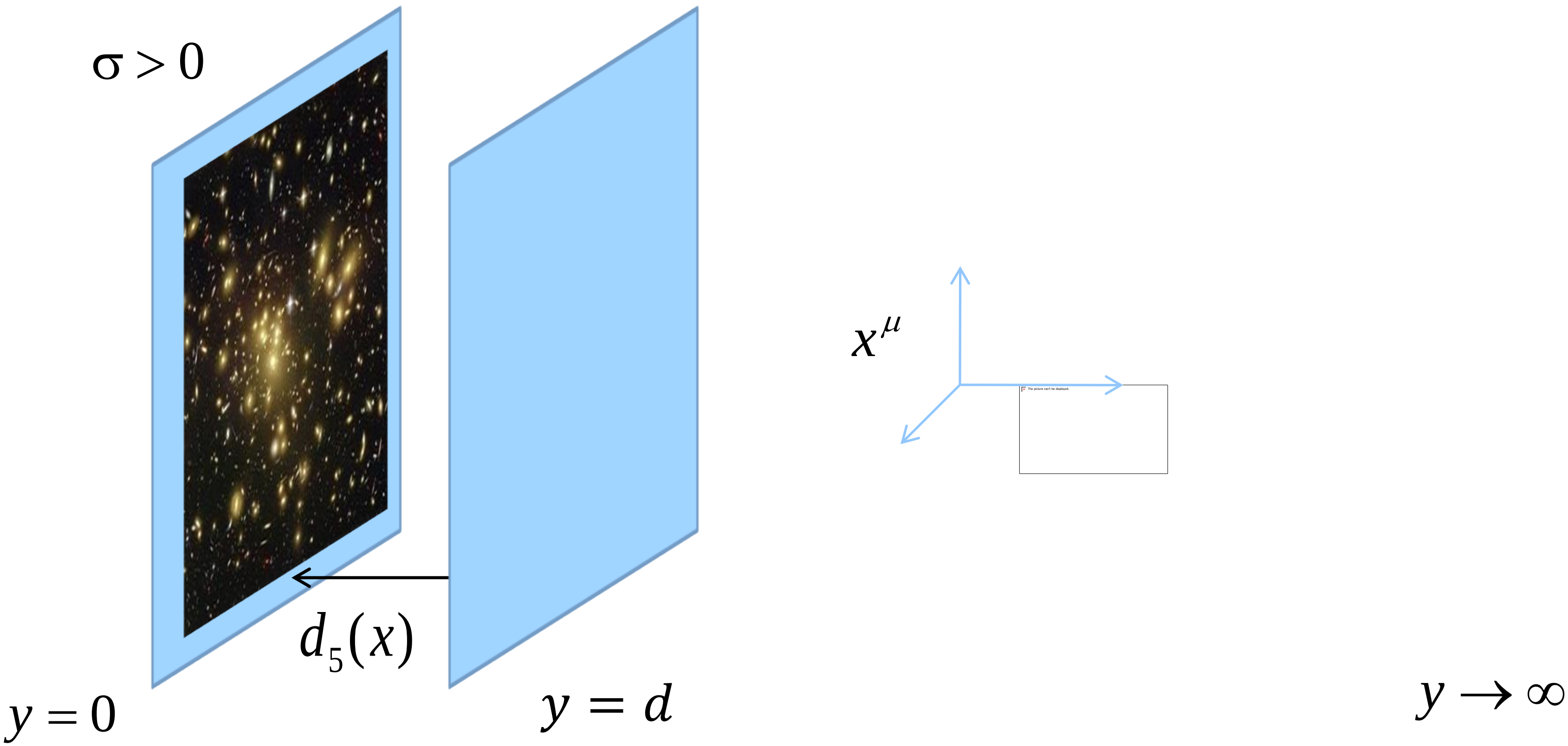
$$m_0,\ k,\ M,\ \mu_c = 1/d$$

Second Randall-Sundrum model (RS II)

As before, the 5-dim bulk is ADS_5 with line element

$$ds^2_{(5)} = G_{ab} dX^a dX^b = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

The observers reside on the positive tension brane at y=0 and the negative tension brane is pushed off to infinity in the fifth dimension



"Sidedness"

In the original RSII model one assumes the Z_2 symmetry

$$z \leftrightarrow z_{\mathrm{br}}/z \qquad \text{or} \qquad y - y_{\mathrm{br}} \leftrightarrow y_{\mathrm{br}} - y$$

so the region $0 < z \le z_{\mathrm{br}}$ is identied with $z_{\mathrm{br}} \le z < \infty$ with the observer brane at the fixed point $z = z_{\mathrm{br}}$. The braneworld is sitting between two patches of AdS_5, one on either side, and is therefore dubbed "two-sided". In contrast, in the "one-sided" RSII model the region $0 < z \le z_{\mathrm{br}}$ is simply cut off.

For simplicity, from now on we will restrict attention to the "one-sided" RSII model. We can place the brane at $z_{\mathrm{br}} = \ell$, which corresponds to $y = 0,$ so in the one-sided regularization we discard the region $y < 0$.

Derivation of the RSII model

See Appendix of N.B., Phys. Rev. D 93, 066010 (2016) arXiv:1511.07323

RS model is a 4+1-dim. universe with AdS_5 geometry containing **two** 3-branes with opposite brane tensions separated in the 5th dimension.

$$S = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{br1}} + S_{\text{br2}}$$

The bulk action is given by

$$S_{\text{bulk}} = \frac{1}{8\pi G_5} \int d^5x \sqrt{\det G} \left(-\frac{R^{(5)}}{2} - \Lambda_5 \right)$$

where Λ_5 is the negative bulk cosmological constant related to the AdS curvature radius as

$$\Lambda_5 = -\frac{6}{\ell^2}$$

The Gibbons-Hawking boundary term is given by an integral over the brane hypersurface Σ

$$S_{\text{GH}} = \frac{1}{8\pi G_5} \int_{\Sigma} d^4x \sqrt{-\det h}\, K$$

The quantity K is the trace of the extrinsic curvature tensor K_{ab} which we have defined as

$$K_{ab} = h_a^c h_b^d n_{d;c}$$

where n_a is a unit vector normal to the brane pointing towards increasing z, h_{ab} is the induced metric

$$h_{ab} = G_{ab} + n_a n_b \qquad a,b = 0,1,2,3,4$$

and h=det $h_{\mu\nu}$ is its determinant, μ,ν=0,1,2,3

The brane action for each brane is given by the Nambu-Goto action

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-\det h_{\mu\nu}}$$

Observers reside on the positive tension brane at y=0. The observer total action (including matter) is

$$S_{\mathrm{br}}\,|_{y=0} = -\sigma \int_\Sigma d^4x \sqrt{-h} + \int_\Sigma d^4x \sqrt{-h}\,\mathcal{L}_{\mathrm{matt}}$$

The Einstein field equations are the bulk field equations

$$R^{(5)}_{ab} - \frac{1}{2} R^{(5)} G_{ab} = \Lambda_5 G_{ab} \qquad (2.4)$$

The junction conditions on the brane are

$$[[K^\mu_\nu - \delta^\mu_\nu K^\alpha_\alpha]] = 8\pi G_5 (\sigma \delta^\mu_\nu + T^\mu_\nu) \qquad (2.5)$$

where the energy momentum tensor $T^\mu{}_\nu$= diag (ρ, -p,-p,-p) describes matter on the brane and [[f]] denotes the discontinuity of a function $f(z)$ across the brane, i.e.,

$$[[f(z)]] = \lim_{\epsilon \to 0} \left(f(z_{\rm br} + \epsilon) - f(z_{\rm br} - \epsilon) \right)$$

Then, for the one-sided regularization, one may use

$$K_{\mu\nu}\big|_{z_{\rm br}+\epsilon} = \frac{8\pi G_5}{3} \left[3T_{\mu\nu} - (\sigma + T) g_{\mu\nu} \right] \qquad (2.6)$$

To derive the RSII model solution, it is convenient to use the Gaussian normal coordinates $x_a = (x_\mu , y)$ with the fifth coordinate y related to the Fefferman-Graham coordinate z by $z = \ell e^{y/\ell}$.
Without loss of generality, we may put observer's brane at $y_{br} = 0$.

We start with a simple ansatz for the line element

$$ds^2_{(5)} = \psi^2(y) g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2 \qquad (2.7)$$

where we assume that $\psi(\infty)=0$ and $\psi(0)=1$. Then one finds the relevant components of the Ricci tensor

$$R^{(5)}_{55} = -4\frac{\psi''}{\psi}, \quad R^{(5)}_{5\mu} = 0,$$

$$R^{(5)}_{\mu\nu} = R_{\mu\nu} + \left(3\psi'^2 + \psi\psi''\right) g_{\mu\nu} \qquad (2.8)$$

and the Ricci scallar

$$R^{(5)} = \frac{R}{\psi^2} + 12\frac{\psi'^2}{\psi^2} + 8\frac{\psi''}{\psi}, \qquad (2.9)$$

where the prime ′ denotes a derivative with respect to y . Using this, the action may be brought to the form

$$S[g] = \frac{1}{8\pi G_5} \int d^4x \sqrt{-g} \int dy \left[-\frac{R}{2}\psi^2 - 4(\psi^3\psi')' \right. \\ \left. + 6\psi^2(\psi')^2 - \Lambda^{(5)}\psi^4 \right] + S_{\rm GH}[g] + S_{\rm br}[g], \qquad (2.10)$$

The extrinsic curvature is easily calculated using the definition and the unit normal vector $n = (0; 0; 0; 0; 1)$. The relevant components

$$K_{\mu\nu} = n_{\mu;\nu} = -\Gamma^{a}_{\mu\nu} n_a = \psi\psi' g_{\mu\nu} \qquad (2.11)$$

The fifth coordinate may be integrated out if $\psi \rightarrow 0$ sufficiently fast as we approach $y = \infty$

The functional form of ψ is found by solving the Einstein equations (2.4) in the bulk. Using the components of the Ricci tensor (2.8) and Ricci scalar (2.9) we obtain

$$6\frac{{\psi'}^2}{\psi^2} + \Lambda_5 + \frac{R}{2\psi^2} = 0 \qquad (2.12)$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \left(3{\psi'}^2 + 3\psi\psi'' + \Lambda_5\psi^2\right) g_{\mu\nu} \qquad (2.13)$$

Combining (2.12) and (2.13) we find (Exercise No 9)

$$\psi = e^{-y/\ell} \qquad \text{where} \qquad \ell = \sqrt{-6/\Lambda_5}$$

With this solution, the metric (2.7) is AdS_5 in Gaussian normal coordinates and Equation (2.13) reduces to the four-dimensional Einstein equation in empty space

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

This equation should follow from the variation of the action (2.10) with $\mathcal{L}_{\text{matt}} = 0$ after integrating out the fifth coordinate. For this to happen it is necessary that the last three terms in square brackets in (2.10) are canceled by the boundary term and the brane action without matter. Using (2.11) one finds that the integral of the second term is canceled by the Gibbons-Hawking term. Then, the integration over y fom 0 to ∞ yields

$$S_{\text{bulk}} + S_{\text{GH}} = \int d^4x \sqrt{-g} \left\{ -\frac{\ell}{32\pi G_5} R + \frac{3}{8\pi \ell G_5} \right\} \quad (2.14)$$

This term contributes to the cosmological constant and can be cancelled by the contribution of the brane action

For the two branes at y=0 and y=l we find

$$g_{\mu\nu}^{\mathrm{ind}}|_{y=0} = g_{\mu\nu} \qquad\qquad g_{\mu\nu}^{\mathrm{ind}}|_{y=d} = e^{-2kd} g_{\mu\nu}$$

The total brane contribution (excluding matter) is

$$S_{\mathrm{br}}|_{y=0} + S_{\mathrm{br}}|_{y=d} = -\sigma \int d^4x\sqrt{-g} - \sigma_d \int d^4x\sqrt{-g}e^{-4kd}$$

This contribution can cancel the last term in (2.14) in the limit $d \to \infty$ provided

$$\sigma = \sigma_0 \equiv \frac{3}{8\pi G_5 \ell} \qquad (2.15)$$

This is the so-called **RSII fine-tuning condition** which assures vanishing of the cosmological constant on the brane. A slight departure from the exact equality $\sigma = \sigma_0$ could, in principle, accommodate the required empirical value of the cosmological constant.

In this way, after integrating out the fifth dimension, the total effective four-dimensional action assumes the form of the standard Einstein-Hilbert action without cosmological constant and without matter

$$S = \frac{1}{8\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} \left(-\frac{R}{2} \right)$$

where G_{N} is the Newton constant defined by

$$\frac{1}{G_{\mathrm{N}}} = \frac{1}{G_5} \int_0^\infty e^{-2ky}\, dy = \frac{1}{2kG_5}$$

This can be written as

$$M_{\mathrm{Pl}}^2 = M^3 \int_0^\infty e^{-2ky}\, dy = \frac{M^3}{2k}$$

Where $M_{\mathrm{Pl}} = G_{\mathrm{N}}^{-1/2}$ and $M \equiv G_5^{-1/3}$ is the 5-dim fundamental scale. In this way, the inverse curvature k serves as the compactification scale and hence the model provides an alternative to compactification.

It may be shown that the fine tuning condition (2.15) follows directly from the junction conditions (2.6)

Exercise No 10: Derive the RSII fine-tuning condition (2.15) from the junction conditions

$$K_{\mu\nu}\big|_{z_{\rm br}+\epsilon} = \frac{8\pi G_5}{3}\left[3T_{\mu\nu} - (\sigma + T)g_{\mu\nu}\right]$$

for a brane without matter at y=0 and the metric

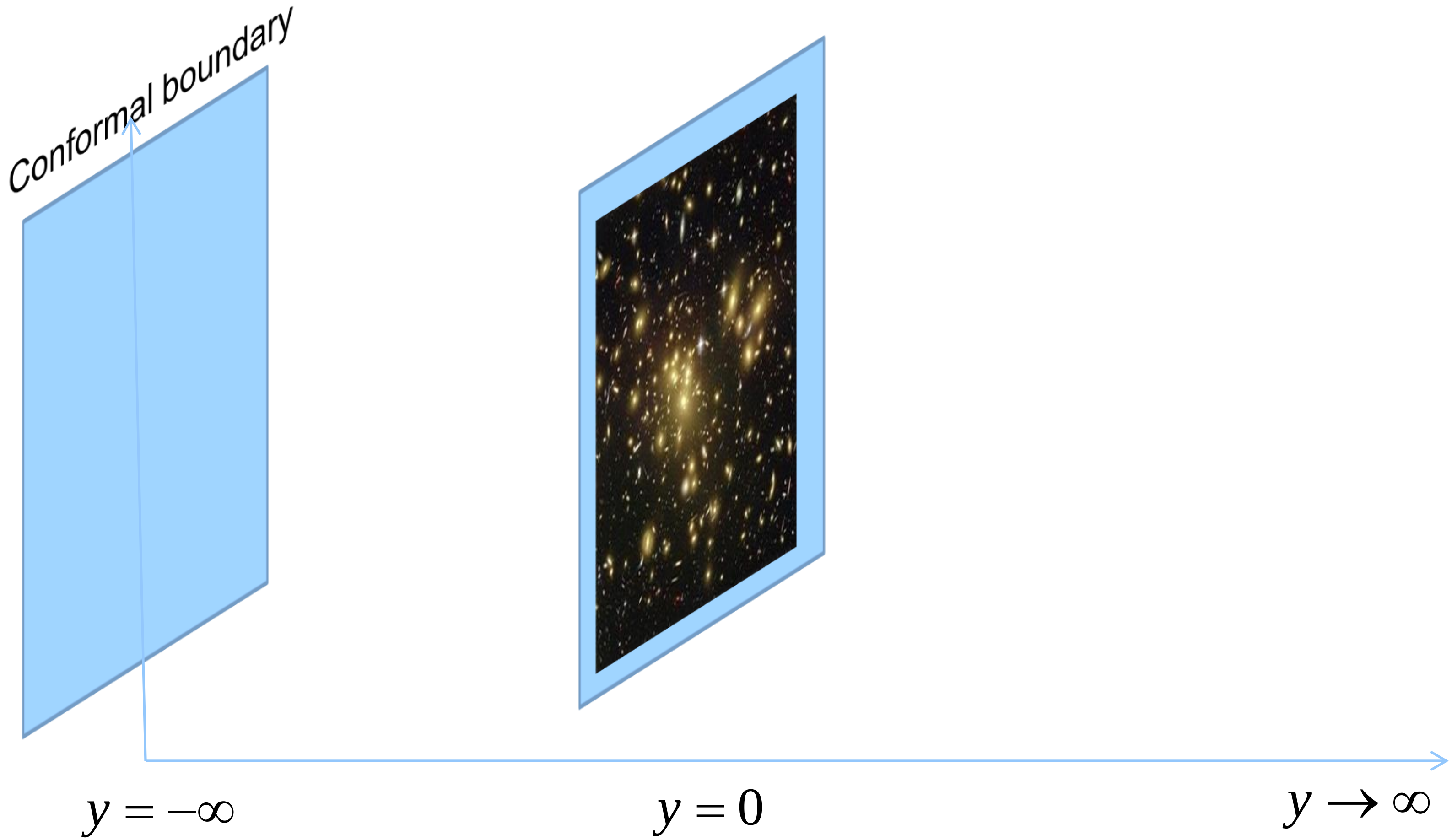
$$ds^2_{(5)} = e^{-2\ell y}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

Hint: use Eq. (2.11)

N.B., PRD **93**, 066010 (2016) , arXiv:1511.07323

RSII Cosmology – Dynamical Brane

Cosmology on the brane is obtained by allowing the brane to move in the bulk. Equivalently, the brane is kept fixed at y=0 while making the metric in the bulk time dependent.



Simple derivation of the RSII braneworld cosmology

J. Soda, Lect. Notes Phys. **828,** 235 (2011) arXiv:1001.1011

See also Appendix in N.B., PRD **93**, 066010 (2016) , arXiv:1511.07323

Consider a time dependent brane hypersurface Σ defined by

$$r - a(t) = 0$$

in AdS-Schwarzschild background. The normal to Σ is

$$n_\mu \propto \partial_\mu (r - a(t)) = (-\partial_t a, 0, 0, 0, 1)$$

Using the normalization $G^{\mu\nu} n_\mu n_\nu = -1$ one finds the nonvanishing components

$$n_t = -\frac{f^{1/2} \partial_t a}{(f^2 - (\partial_t a)^2)^{1/2}} \qquad n_r = \frac{f^{1/2}}{(f^2 - (\partial_t a)^2)^{1/2}}$$

where

$$f(a) = \frac{a^2}{\ell^2} + \kappa - \mu \frac{\ell^2}{a^2} \tag{2.16}$$

Then, the induced line element on the brane is

$$ds_{\rm ind}^2 = n^2(t)dt^2 - a^2(t)d\Omega_k^2$$

where $$n^2 = f(a) - \frac{(\partial_t a)^2}{f(a)} \qquad d\Omega_\kappa^2 = d\chi^2 + \frac{\sin^2\sqrt{\kappa}\chi}{\kappa}d\Omega^2$$

The junction conditions on the brane with matter may be written as

$$K_{\mu\nu}|_{r=a-\epsilon} = -\frac{8\pi G_5}{3}\left[(\sigma+T)g_{\mu\nu} - 3T_{\mu\nu})\right]$$

The $\chi\chi$-component gives

$$\frac{f^{3/2}}{(f^2-(\partial_t a)^2)^{1/2}} = \frac{8\pi G_5}{3}(\sigma+\rho)a$$

Exercise No 11: Derive the $\chi\chi$ -component of the junction condition

This may be written as

$$\frac{(\partial_t a)^2}{n^2 a^2} + \frac{f}{a^2} = \frac{1}{\ell^2 \sigma_0^2}(\sigma + \rho)^2$$

Hubble expansion rate on the brane

Substituting for *f* the expression (2.16) we obtain

$$H_{\mathrm{RSII}}^2 + \frac{\kappa}{a^2} = \frac{(\sigma + \rho)^2}{\ell^2 \sigma_0^2} - \frac{1}{\ell^2} + \frac{\mu \ell^2}{a^4}$$

where

$$H_{\mathrm{RSII}}^2 = \frac{(\partial_t a)^2}{n^2 a^2}$$

Employing the RSII fine tuning condition (2.15)

$$\sigma = \sigma_0 \equiv \frac{3}{8\pi G_5 \ell} = \frac{3}{4\pi G_{\mathrm{N}} \ell^2}$$

we find the effective Friedmann equation

$$H_{\mathrm{RSII}}^2 + \frac{\kappa}{a^2} = \frac{8\pi G_{\mathrm{N}}}{3}\rho + \boxed{\left(\frac{4\pi G_{\mathrm{N}}\ell}{3}\right)^2 \rho^2 + \frac{\mu\ell}{a^4}} \qquad (2.17)$$

Quadratic deviation from the standard FRW. Decays rapidly as $\sim a^{-8}$ in the radiation epoch

dark radiation

due to a black hole in the bulk – should not exceed 10% of the total radiation content in the epoch of BB nucleosynthesis

RSII cosmology is thus subject to astrophysical tests

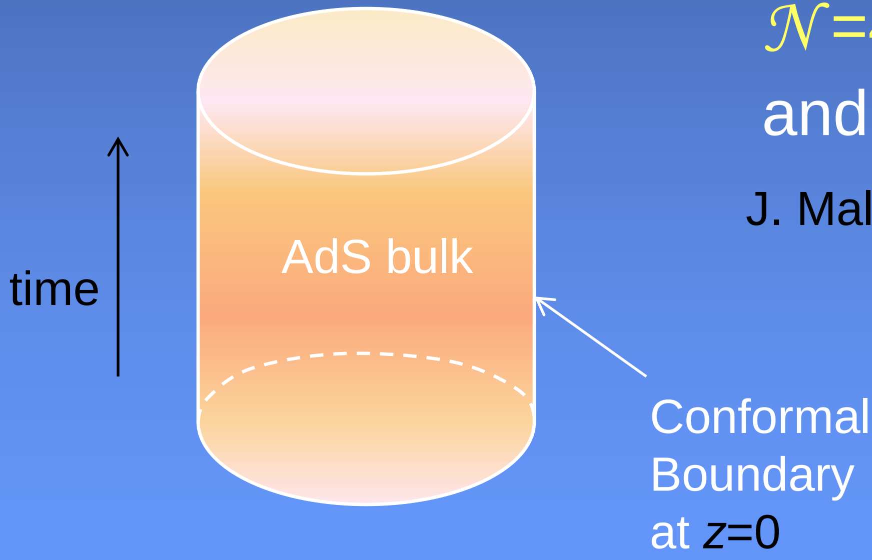
Thank you

AdS/CFT and Braneworld Holography

AdS/CFT correspondence is a holographic duality between gravity in d+1-dim space-time and quantum CFT on the d-dim boundary. Original formulation stems from string theory:

Equivalence of 3+1-dim
$\mathcal{N}$=4 Supersymmetric YM Theory
and string theory in $AdS_5 \times S_5$

J. Maldacena, Adv. Theor. Math. Phys. **2** (1998)



Examples of CFT:
quantum electrodynamics,
Yang Mills gauge theory,
massless scalar field theory,
massless spin ½ field theory

Why AdS?

Anti de Sitter space is a maximally symmetric solution to Einstein's equations with negative cosmological constant.
In 4+1 dimensions the symmetry group is $AdS_5 \equiv$ **SO(4,2)**

The bulk metric may be represented by (Fefferman-Graham coordinates)

$$ds^2_{(5)} = G_{ab} dX^a dX^b = \frac{\ell^2}{z^2}(g_{\mu\nu} dx^\mu dx^\nu - dz^2)$$

So there is a boundary at z=0. A correspondence between gravity in the bulk and the conformal field theory (CFT) on the boundary of AdS may be expected because the 3+1 boundary conformal field theory is invariant under conformal transformations: Poincare + dilatations + special conformal transformation = conformal group ≡ **SO(4,2)**

It is sometimes convenient to represented the metric in Gaussian normal coordinates

$$ds^2_{(5)} = e^{-2ky} g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2$$

Warp factor

Spherical Fefferman Graham coordinates are obtained from (2.2) by defining the coordinete z via

$$\frac{dz}{z} = -\frac{dr}{\ell\sqrt{f}}$$

This gives (with an appropriate integration constant)

$$\frac{z^4}{\ell^4} = \frac{16}{\kappa^2 + 4\mu}\,\frac{r^2 + \frac{\kappa}{2} - r\sqrt{f}}{r^2 + \frac{\kappa}{2} + r\sqrt{f}}$$

The inverted relation can be written as

$$r^2 = \frac{\alpha + \beta z^2 + \gamma z^4}{z^2} \qquad (23)$$

where $\alpha = \ell^4 \qquad \beta = -\kappa\ell^2 \qquad \gamma = \dfrac{\kappa^2 + 4\mu}{16}$

Consider a 5-dim bulk action in AdS$_5$ background

$$S_{(5)}[\Phi] = \int d^5x \sqrt{-G}\,\mathcal{L}_{(5)}\left(\Phi, G_{ab}\right)$$

The bulk field Φ is completely determined by its field equations obtained from the variational principle

$$\frac{\delta S_{(5)}}{\delta \Phi} = 0$$

given the boundary value $\varphi(x) \equiv \Phi(z=0,x)$ and the induced metric on the boundary $h_{\mu\nu}$.

Using the solution $\Phi = \Phi[\varphi, h]$ we can define a functional

$$S[\varphi, h] = S^{\text{shell}}\big[\Phi[\varphi, h]\big]$$

where $S^{\text{shell}}\big[\Phi[\varphi, h]\big]$ is the **on-shell** bulk action, i.e., the action in which the fields are solutions of the equations of motion given their boundary values. The on-shell bulk action is still subject to the variation of the boundary values.

AdS/CFT conjecture: The action $S[\varphi,h]$ can be identified with the generating functional of a conformal field theory on the boundary

$$S[\varphi,h] \equiv \ln \int d\psi \exp\left\{-\int d^4x\sqrt{-h}\left[\mathcal{L}^{\mathrm{CFT}}(\psi(x)) - O(\psi(x))\varphi(x)\right]\right\}$$

where the boundary fields serve as sources for CFT operators

$\mathcal{L}^{\mathrm{CFT}}(\psi)$ – conformal field theory Lagrangian

$O(\psi)$ – operators of dimension Δ

In this way the CFT correlation functions can be calculated as functional derivatives of the on-shell bulk action, e.g.,

$$\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} = \left\langle O(\psi(x))O(\psi(y))\right\rangle - \left\langle O(\psi(x))\right\rangle\left\langle O(\psi(y))\right\rangle$$

Consider a bulk action with **only gravity** in the bulk

$$S = \frac{1}{8\pi G_5}\int d^5x\sqrt{-G}\left(-\frac{R^{(5)}}{2}-\Lambda_5\right)$$

Given induced metric $h_{\mu\nu}$ on the boundary the geometry is completely determined by the field equations obtained from the variation principle

$$\frac{\delta S}{\delta G_{ab}} = 0$$

yielding a solution $G_{ab}[h]$. Using this we define a functional

$$S[h] = S^{\,\mathrm{shell}}[G_{ab}[h]]$$

where $S^{\mathrm{shell}}[G_{ab}[h]]$ is the on-shell bulk action

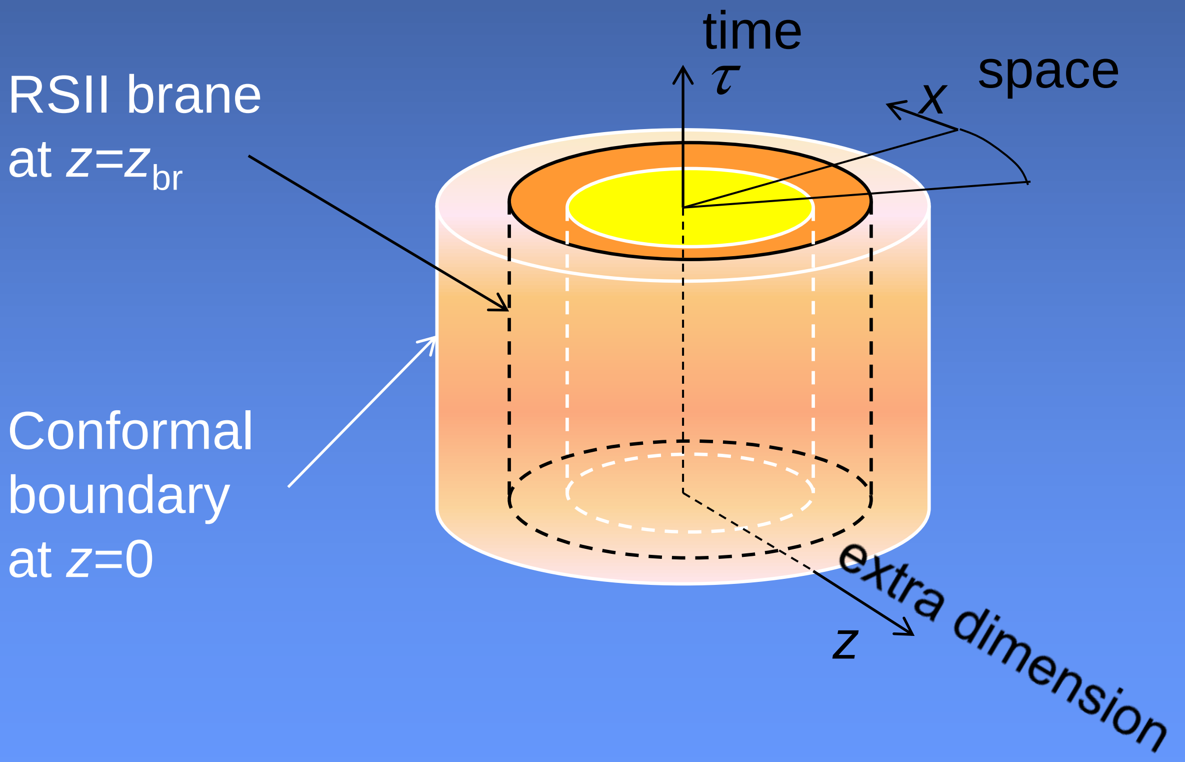
AdS/CFT conjecture: As before, the action $S[h]$ can be identified with the generating functional of a conformal field theory (CFT) on the boundary.

The induced metric $h_{\mu\nu}$ serves as the source for the **energy-momentum tensor** of the dual CFT so that its vacuum expectation value is obtained from the on shell classical action

$$\left\langle T_{\mu\nu}^{\mathrm{CFT}} \right\rangle = \frac{1}{2\sqrt{-h}} \frac{\delta S}{\delta h^{\mu\nu}}$$

In the second Randall-Sundrum (RS II) model a 3-brane is located at a finite distance from the boundary of AdS_5.

Foliation of the bulk:



In the RSII model by introducing the boundary in AdS_5 at $z = z_{br}$ instead of $z = 0$, the model is conjectured to be dual to a cutoff CFT coupled to gravity, with $z = z_{br}$ providing the IR cutoff (corresponding to the UV catoff of the boundary CFT)

Holographic renormalization

The on-shell bulk action is IR divergent because physical distances diverge at z=0

$$ds^2_{(5)} = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2)$$

A 4-dim asymptotically AdS metric near z=0 can be expanded as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots$$

Explicit expressions for $g^{(2n)}_{\mu\nu},\ n = 2,4$, in terms of arbitrary $g^{(0)}_{\mu\nu}$

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}R g^{(0)}_{\mu\nu}\right) \qquad \mathrm{Tr}\, g^{(4)} = -\frac{1}{4}\mathrm{Tr}\,(g^{(2)})^2$$

de Haro, Solodukhin, Skenderis, Comm. Math. Phys. **217** (2001)

We regularize the action by placing a brane (RSII brane) near the AdS boundary, i.e., at $z = \varepsilon\ell$, $\varepsilon << 1$, so that the induced metric on the brane is

$$h_{\mu\nu} = \frac{1}{\varepsilon^2}\left(g^{(0)}_{\mu\nu} + \varepsilon^2 \ell^2 g^{(2)}_{\mu\nu} + \cdots\right)$$

The bulk splits in two regions: $0 \leq z \leq \varepsilon\ell$, and $\varepsilon\ell \leq z \leq \infty$. We can either discard the region $0 \leq z \leq \varepsilon$ (one-sided regularization) or invoke the Z_2 symmetry (as in the original RSII model) and identify two regions (two-sided regularization). As before, we use the one-sided regularization.

The regularized bulk action is given by

$$S^{\mathrm{reg}}[h] = \frac{1}{8\pi G_5} \int_{z \geq \varepsilon\ell} d^5x \sqrt{-G} \left(-\frac{R^{(5)}}{2} - \Lambda_5 \right) + S_{\mathrm{GH}}[h] + S_{\mathrm{br}}[h]$$

$h_{\mu\nu}$ – induced metric on the brane

$S_{\mathrm{GH}}[h]$ – Gibbons-Hawking boundary term

$S_{\mathrm{br}}[h] = -\sigma \int d^4x \sqrt{-h} + \int d^4x \sqrt{-h}\, \mathcal{L}_{\mathrm{matt}}$ – brane action

The equations of motion on the brane are obtained by demanding that the variation with respect to the induced metric $h^{\mu\nu}$ of the regularized on shell bulk action (RSII action) vanishes, i.e.,

$$\delta S^{\mathrm{reg}}[h] = 0$$

Next we renormalize the boundary action. The renormalized boundary action is obtained by adding counter-terms and taking the limit ε→0

$$S^{\mathrm{reg}}[h] = \lim_{\varepsilon \to 0}(S^{\mathrm{ren}}[h] + S_1[h] + S_2[h] + S_3[h])$$

The necessary counter-terms are

$$S_1[h] = -\frac{6}{16\pi G_5 \ell} \int d^4x \sqrt{-h},$$

$$S_2[h] = -\frac{\ell}{16\pi G_5} \int d^4x \sqrt{-h} \left(-\frac{R[h]}{2} \right),$$

$$S_3[h] = -\frac{\ell^3}{16\pi G_5} \int d^4x \sqrt{-h} \frac{\log \epsilon}{4} \left(R^{\mu\nu}[h] R_{\mu\nu}[h] - \frac{1}{3} R^2[h] \right)$$

S.W. Hawking, T. Hertog, and H.S. Reall, Phys. Rev. D **62** (2000)

Now we demand that the variation with respect to the induced metric $h^{\mu\nu}$ of the regularized on shell bulk action (RSII action) vanishes, i.e.,

$$\delta S^{\text{reg}}[h] = 0$$

This may be expressed as

matter on the brane

$$\delta\left[S^{\text{ren}} - S_3 - \left(\sigma - \frac{3}{8\pi \ell G_5} \right) \int d^4x \sqrt{-h} + \int d^4x \sqrt{-h}\,\mathcal{L}_{\text{matt}} + \frac{\ell}{16\pi G_5} \int d^4x \sqrt{-h}\, \frac{R}{2} \right] = 0.$$

cosmological Constant term

Einstein Hilbert term

AdS/CFT prescription

$$\delta(S^{\text{ren}} - S_3) = \frac{1}{2} \int d^4x \sqrt{-h} \left\langle T_{\mu\nu}^{\text{CFT}} \right\rangle \delta h^{\mu\nu}$$

The variation of the scheme-dependent S_3 combined with S^{ren} yields

$$\frac{2}{\sqrt{-h}}\frac{\delta(S^{\mathrm{ren}}-S_3)}{\delta h^{\mu\nu}}=\left\langle T_{\mu\nu}^{\mathrm{CFT}}\right\rangle$$

where

$$\langle T_{\mu\nu}^{\mathrm{CFT}}\rangle=-\frac{\ell^3}{4\pi G_5}\left\{g_{\mu\nu}^{(4)}-\frac{1}{8}\left[(\mathrm{Tr}g^{(2)})^2-\mathrm{Tr}(g^{(2)})^2\right]g_{\mu\nu}^{(0)}-\frac{1}{2}(g^{(2)})^2_{\mu\nu}+\frac{1}{4}\mathrm{Tr}g^{(2)}g_{\mu\nu}^{(2)}\right\}$$

and $g_{\mu\nu}^{(2n)}$ are the coefficients that appear in the expansion of the bulk metric

de Haro et al, Comm. Math. Phys. **217** (2001)

This is an explicit realization of the AdS/CFT correspondence:
the vacuum expectation value of a boundary CFT operator is obtained in terms of geometrical quantities of the bulk.

The variation of the action yields Einstein's equations on the boundary

$$R_{\mu\nu} - \frac{1}{2} R g^{(0)}_{\mu\nu} = 8\pi G_{\mathrm{N}} \left(\left\langle T^{\mathrm{CFT}}_{\mu\nu} \right\rangle + T^{\mathrm{matt}}_{\mu\nu} \right)$$

metric on the boundary

Holographic cosmology

We now seek a cosmological solution to the holographic Einstein equations such that the induced metric at the boundary has the FRW form

$$ds^2 = g^{(0)}_{\mu\nu} dx_\mu dx_\nu = dt^2 - a^2(t) d\Omega_k^2$$

First, we represent the bulk metric in AdS-Schwarzschild static coordinates $(\tau, r, \chi, \vartheta, \varphi)$

$$ds^2_{(5)} = f(r) d\tau^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_\kappa^2$$

where

$$f(r) = \frac{r^2}{\ell^2} + \kappa - \mu \frac{\ell^2}{r^2} \qquad \mu = \frac{8 G_5 M_{\mathrm{bh}}}{3\pi \ell^2}$$

$$d\Omega_\kappa^2 = d\chi^2 + \frac{\sin^2(\sqrt{\kappa}\chi)}{\kappa}(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

Starting from AdS-Schwarzschild static coordinates
and making the coordinate transformation $\tau = \tau(t, z),\ r = r(t, z)$
the bulk line element will take a general form

$$ds_{(5)}^2 = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2) = \frac{\ell^2}{z^2}\left[\mathcal{N}^2(t,z)dt^2 - \mathcal{A}^2(t,z)d\Omega_k^2 - dz^2\right]$$

Imposing the boundary conditions at z=0:

$$\mathcal{N}(t,0) = 1, \quad \mathcal{A}(t,0) = a(t)$$

the induced metric at the boundary takes the FRW form

$$ds^2 = dt^2 - a^2(t)d\Omega_k^2$$

From the Einstein equations at the boundary we obtain the holographic Friedmann equation (from now on we assume spatial flatness, i.e., we put $\kappa = 0$)

$$H^2 - \frac{\ell^2}{4} H^4 = \frac{8\pi G_\mathrm{N}}{3}\rho + \frac{4\mu\ell^2}{a^4}$$

quadratic deviation

dark radiation

The second Friedmann equation can be derived from the energy-momentum conservation

$$\dot{H}\left(1 - \frac{\ell^2}{2} H^2\right) = -4\pi G_\mathrm{N}(p+\rho) - \frac{2\mu\ell^2}{a^4}$$

quadratic deviation

dark radiation

where $\rho = T_{00}^{\mathrm{matt}}, \quad p = -T^{\mathrm{matt}\,i}{}_{i}$

E. Kiritsis, JCAP **0510** (2005); Apostolopoulos et al, Phys. Rev. Lett. **102**, (2009);
N.B., Phys. Rev. D 93 (2016), arXiv:1511.07323

The holographic cosmology has interesting properties. Solving the first Friedmann equation as a quadratic equation for H^2 we find

$$H^2 = \frac{2}{\ell^2}\left(1 \pm \sqrt{1 - 8\pi\ell^2 G_{\mathrm{N}}\rho/3}\right)$$

Demanding that this equation reduces to the standard Friedmann equation in the low energy limit, i.e., in the limit when

$$\ell^2 G_{\mathrm{N}}\rho \ll 1$$

it follows that we must discard the + sign solution. Then, it follows that the physical range of the Hubble rate is between 0 and $\sqrt{2}/\ell$ starting from its maximal value $H_{\max} = \sqrt{2}/\ell$ at an arbitrary initial time t_0. At that time, which may be chosen to be zero, the density and cosmological scale are both finite so the Big-Bang singularity is avoided!

Thank you

Conformal anomaly

$$\left\langle T^{\mathrm{CFT}\,\mu}{}_{\mu} \right\rangle = \frac{\ell^3}{128\pi G_5}\left(G_{\mathrm{GB}} - C^2\right)$$

$$G_{\mathrm{GB}} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$$ Gauss-Bonnet invariant

$$C^2 \equiv C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2$$ Weyl tensor squared

compared with the general result from field theory

$$\left\langle T^{\mathrm{CFT}\,\mu}{}_{\mu} \right\rangle = bG_{\mathrm{GB}} - cC^2 + b'\Box R$$

The two results agree if we ignore the last term and identify

$$b = c = \frac{\ell^3}{128\pi G_5}$$

Generally $b \neq c$ because

$$b = \frac{n_\mathrm{s} + (11/2)n_\mathrm{f} + 62n_\mathrm{v}}{360(4\pi)^2} \qquad c = \frac{n_\mathrm{s} + 3n_\mathrm{f} + 12n_\mathrm{v}}{120(4\pi)^2}$$

but in the $\mathcal{N} = 4$ U(N) super YM theory $b=c$ with

$$n_\mathrm{s} = 6N^2, n_\mathrm{f} = 4N^2, n_\mathrm{v} = N^2$$

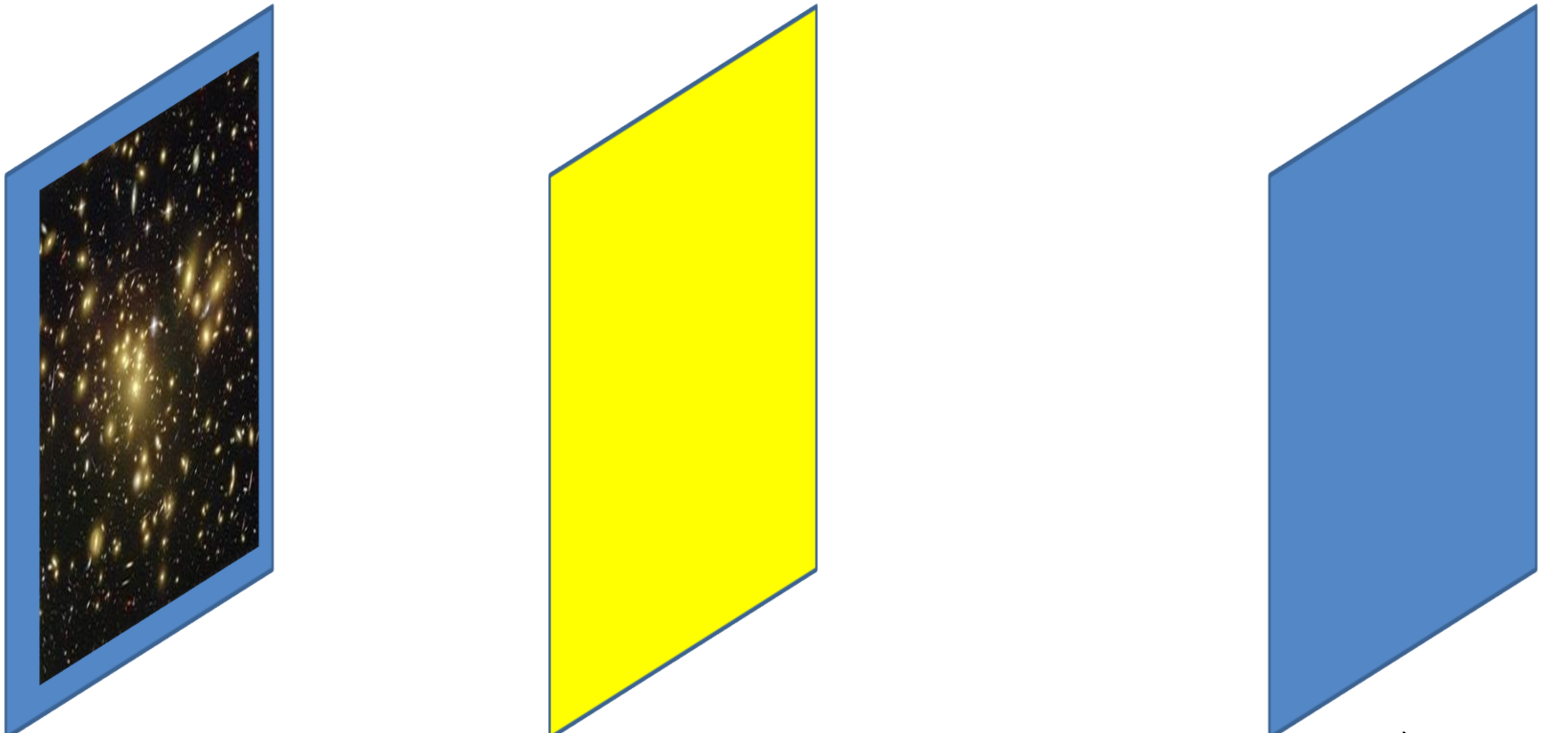
The conformal anomaly is correctly reproduced if we identify

$$\frac{\ell^3}{G_5} = \frac{2N^2}{\pi}$$

Dynamical Brane as a Tachyon

Consider an additional 3-brane moving in the 5-d bulk spacetime with metric

$$ds^2_{(5)} = \psi^2(y) g^{\mu\nu} dx^\mu dx^\nu - dy^2$$



$y = 0$

$y \to \infty$

The points on the brane can be are parameterized by $X^M = \left(x^\mu, Y\left(x\right)\right)$
The 5-th coordinate *Y* is treated as a dynamical field that depends on *x*.
From the brane action

$$S_{\mathrm{br}} = -\sigma \int d^4 x \sqrt{-\det g_{\mu\nu}^{ind}}$$

using the induced metric

$$g_{\mu\nu}^{\mathrm{ind}} = g_{(5)MN} X_{,\mu}^{M} X_{,\nu}^{N} = \psi^2 (Y) g_{\mu\nu} - Y_{,\mu} Y_{,\nu} \qquad (35)$$

one finds

$$S_{\mathrm{br}} = -\sigma \int d^4 x \sqrt{-g}\, \psi^4 (Y) \left(1 - \psi^{-2} g^{\mu\nu} Y_{,\mu} Y_{,\nu}\right)^{1/2}$$

(36)

Exercise No 12:

(a)Prove the following relation

$$\det(g_{\mu\nu} - \alpha^2 u_\mu u_\nu) = (1-\alpha^2)\det g_{\mu\nu}$$

for a general metric $g_{\mu\nu}$, unit timelike vector u_μ, and $\alpha^2 < 1$

Hint: use a comoving reference frame.

(b) Use (a) to derive (36) from the brane action (35)