Introduction to cosmology in the braneworld

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Lecture 1 – Preliminaries

Basics of the standard cosmology
Legendre transformation and applications
Basic fluid mechanics
Lagrangian and Hamiltonian

Lecture 2 – Braneworld Universe

Hypersurfaces
Strings and branes
Randall-Sundrum model
Braneworld cosmology

Lecture 3 – Holographic Universe and inflation

AdS/CFT and braneworld holography
Holographic cosmology
***k*-essence inflation**

Notation

in 3+1 dimensions:

Greek indices $\mu, \nu, .. = 0,1,2,3,$

$g_{\mu\nu}$ – 4-dim. metric tensor, metric signature is $+ - - \,-$

In 4+1 dimensions:

Latin indeces a,b, =0,1,2,3,4 and

G_{ab} – 5-dim. metric tensor, metric signature is $+ - - - \,-$

$R^{(5)}_{ab}$ – 5-dim. Ricci curvature tensor

$R^{(5)} \equiv G^{ab} \; R^{(5)}_{ab}$ – Ricci scalar

Λ_5 – 5-dim. cosmological constant

G_5 – 5-dim. gravitation constant

Notation

We use the Landau-Lifshitz curvature convention

L.D.Landau, E.M. Lifshitz, *Classical theory of fields*

The Riemann tensor is defined as

$$R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de}$$

and the Ricci tensor as

$$R_{ab} = R^s{}_{asb}$$

Then, the Einstein equations are

$$R_{ab} - \frac{1}{2} R G_{ab} = +8\pi G T_{ab}$$

Lecture 1 – Preliminaries

Basics of the standard cosmology
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Basics of the Standard cosmology

Theoretical pillars:

- **General relativity**

- **Cosmological principle** – **homogeneity** (matter density same everywhere) and **isotropy** (no preferred direction) of space – approximate property on very large scales (~Glyrs today)

- **Fluctuations of geometry** in the early Universe cause structure formation (stars, galaxies, clusters ...)

General Relativity

Gravity is described by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\mathrm{N}} T_{\mu\nu} + g_{\mu\nu} \Lambda$$

where

$g_{\mu\nu}$ - metric tensor

$R_{\mu\nu}$ - Ricci curvature tensor

$R = g^{\mu\nu} R_{\mu\nu}$ - Ricci scalar

Λ - cosmological constant

$T_{\mu\nu}$ - energy – momentum tensor

Cosmological principle

Homogeneity and isotropy of space

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]$$

$a(t)$ – cosmological scale

the curvature constant k takes on the values $1/r_0^2$, 0, or $-1/r_0^2$, for a closed, flat, or open universe, respectively. The above metric is known as the Friedmann-Robertson-Walker (FRW) metric.

Expanding Spacetime

The FRW **metric** can be written in various forms:

Standard representation

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

where

t – synchronous or cosmic time

r – *comoving* radial coordinate

The transformation

$$r = \frac{1}{\sqrt{k}} \sin\sqrt{k}\,\chi$$

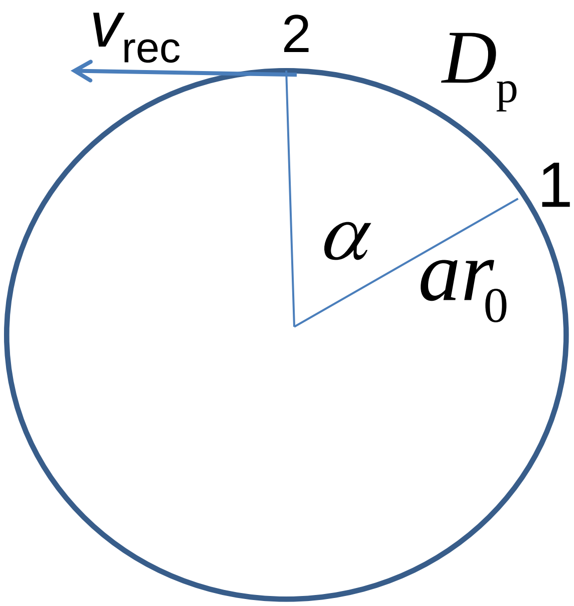
brings the line element to another convenient form

$$ds^2 = dt^2 - a^2(d\chi^2 + \sin^2(\sqrt{k}\,\chi)d\Omega^2)$$

χ –new comoving radial coordinate

Cosmological distances:

Infinitesimal ***proper interval*** is the line ellement $ds_{\mathrm{p}} = \sqrt{-ds^2}$ defined for $dt = 0, d\Omega = 0$



$$ds_{\mathrm{p}} = \frac{adr}{\sqrt{1-kr^2}} = ar_0 d\alpha$$

where $k = 1/r_0^2$, r_0 is the curvature radius, and $\alpha = \chi/r_0$

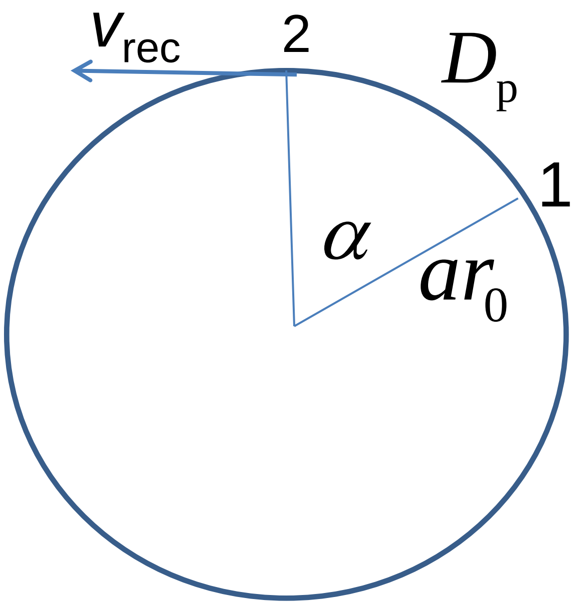
physical or ***proper distance***

$$D_{\mathrm{p}} = \int_1^2 ds_{\mathrm{p}} = ar_0\alpha$$

comoving distance

$$D = \frac{D_{\mathrm{p}}}{a} = r_0\alpha$$

Recession velocity



$$v_{\mathrm{rec}} \equiv \frac{dD_{\mathrm{p}}}{dt} = \dot{a}D = \frac{\dot{a}}{a} D_{\mathrm{p}}$$

We define the Hubble expansion rate as

$$H = \frac{\dot{a}}{a}$$

In this way we obtain Hubble's law

$$v_{\mathrm{rec}} = H D_{\mathrm{p}}$$

Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G_N}{3}(\rho + 3p)$$

$H \equiv \frac{\dot{a}}{a}$ expansion rate

$k \begin{cases} > 0 & \text{closed} \\ = 0 & \text{flat} \\ < 0 & \text{open} \end{cases}$

$\dot{\rho} + 3H(\rho + p) = 0$ ⇒ $\rho \propto a^{-3(1+w)}$ $w = p/\rho$

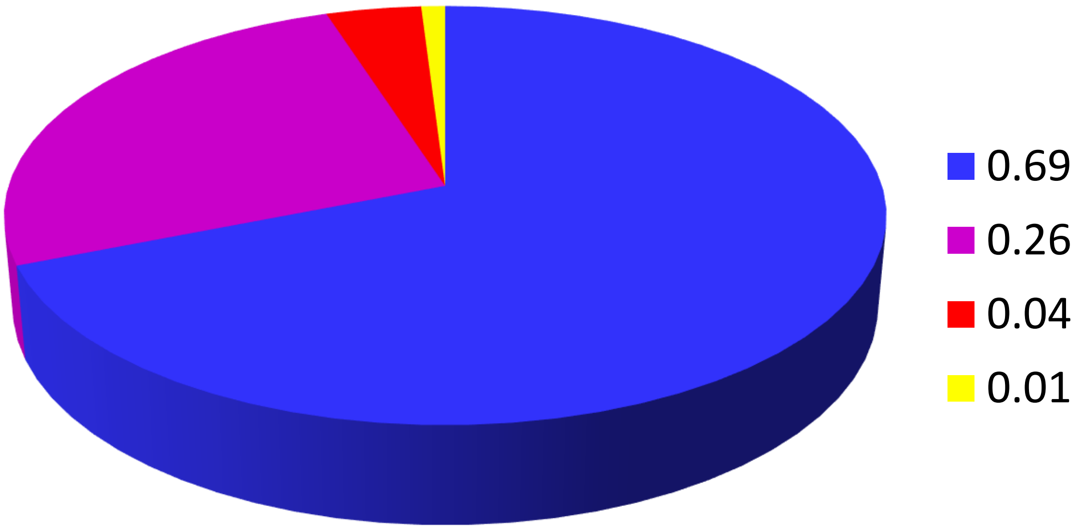
Various kinds of cosmic fluids with linear **equations of state** $p = w\rho$

- radiation $p_R = \rho_R/3$ $w = 1/3$ $\rho \propto a^{-4}$
- dust $p_M = 0$ $w = 0$ $\rho \propto a^{-3}$
- vacuum $p_\Lambda = -\rho_\Lambda$ $w = -1$ $\rho \propto a^0$

Dark nature of the Universe

According to recent observations (Planck Satellite Mission 2018):

More than 99% of matter is not luminous

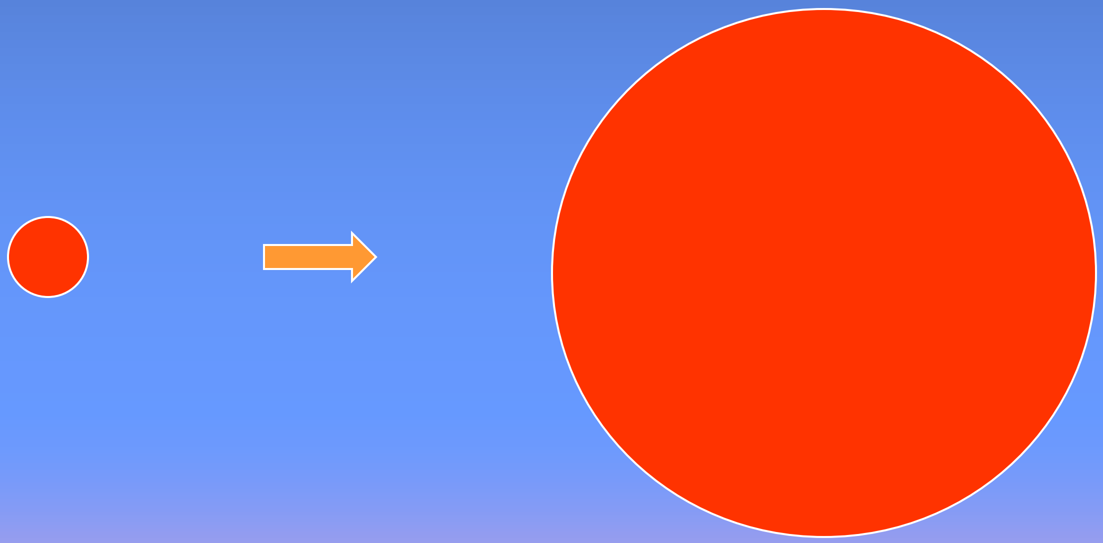


- Out of that less than 5% is ordinary ("baryonic")
- About 26% is Dark Matter
- About 69% is Dark Energy (Vacuum Energy)

Early Universe - Inflation

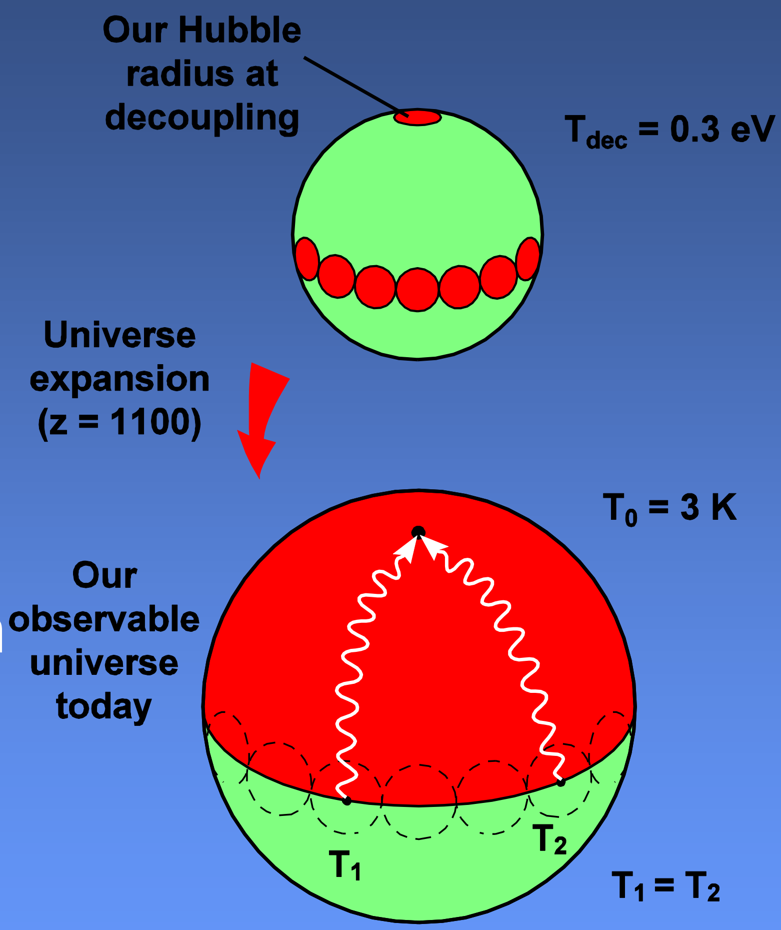
A short period of **inflation follows** – very rapid expansion -

10^{25} times in 10^{-32} s.



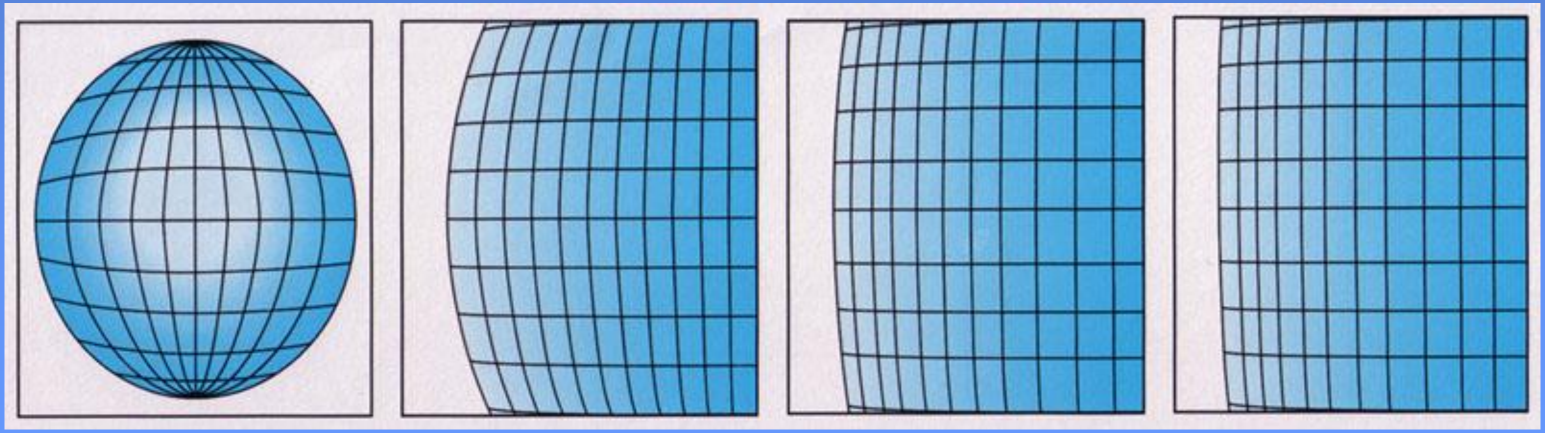
The horizon problem.

Observations of the cosmic microwave background radiation (CMB) show that the Universe is homogeneous and isotropic. The problem arises because the information about CMB radiation arrive from distant regions of the Universe which were not in a causal contact at the moment when radiation had been emitted – in contradiction with the observational fact that the measured temperature of radiation is equal (up to the deviations of at most about 10^{-5}) in all directions of observation.



The flatness problem

Observations of the average matter density, expansion rate and fluctuations of the CMB radiation show that the Universe is flat or with a very small curvature today. In order to achieve this, a “fine-tuning“ of the initial conditions is needed, which is rather unnatural. The answer is given by inflation:



10^{-5} $\gamma = Hd$,

The initial density perturbations

The question is how the initial deviations from homogeneity of the density are formed having in mind that they should be about 10^{-5} in order to yield today's structures (stars, galaxies, clusters). The answer is given by inflation: perturbations of density are created as quantum fluctuations of the inflaton field.

Legendre Transformation

Consider an arbitrary smooth function $f(x)$. We can define another function $g(u)$ such that

$$f(x) + g(u) = xu \qquad (1.1)$$

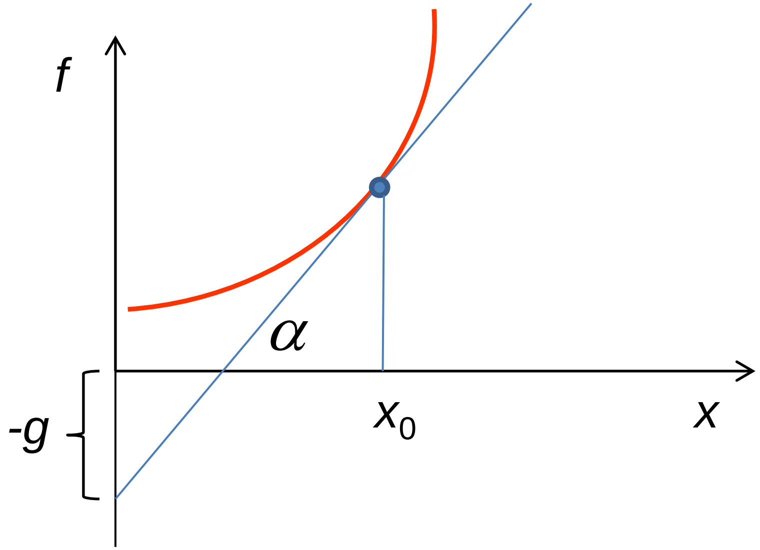
where the variables x and u (called **conjugate variables**) are related via

$$u = \frac{\partial f}{\partial x}, \qquad x = \frac{\partial g}{\partial u}$$

The proof is based on the property that a function $f(x)$ at an arbitrary point x_0 , can be locally represented by

$$f(x_0) = u_0 x_0 - g \qquad \text{where} \qquad u_0 = \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

Simple geometric meaning



$$u_0 = \tan\alpha$$

g depends on u_0

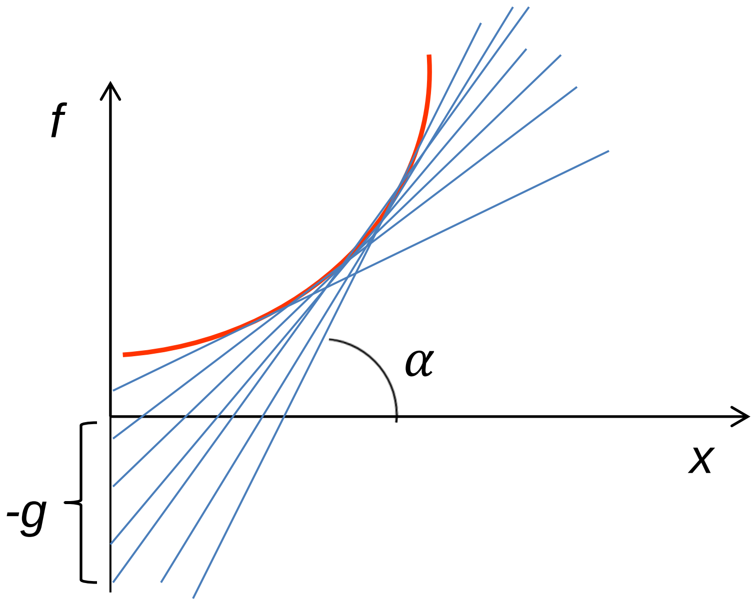
By the symmetry of (1) we can also write

$$g(u_0) = x_0 u_0 - f$$

where

$$x_0 = \left.\frac{\partial g}{\partial u}\right|_{u_0}$$

Exercise No 1: Prove this using the geometry in the figure



By variing x_0 the function $f(x)$ may be regarded as the envelope of tangents

g is an implicit function of $u = \tan\alpha$

The generalization to *n* dimensions is straightforward:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} u_i x_i - g(u_1, u_2, \ldots, u_n)$$

with

$$u_i = \frac{\partial f}{\partial x_i}, \qquad x_i = \frac{\partial g}{\partial u_i}$$

Applications

1) Classical mechanics

$$\mathrm{H}(q_i, p_j) = \sum_j \dot{q}_j p_j - \mathrm{L}(q_i, \dot{q}_j) \qquad p_i = \frac{\partial \mathrm{L}}{\partial \dot{q}_i}, \qquad \dot{q}_i = \frac{\partial \mathrm{H}}{\partial p_i}$$

2) Quantum field theory

$$\Gamma[\phi] = W[J] - \int dx J(x)\phi(x)$$

$$J(x) = -\frac{\delta\Gamma}{\delta\phi(x)}, \qquad \phi(x) = \frac{\delta W}{\delta J(x)}$$

W – generating functional of connected Green's functions

Γ – generating functional of 1-particle irreducible Green's functions

3) Thermodynamics

Canonical ensemble (particle number fixed)

The canonical (Helmholz) **free energy** F as a function of temperature T (and volume V) is related to the internal energy E via a one-dimensional Legendre transformation.

$$F(T) = E(S) - TS$$

with

$$S = -\frac{\partial F}{\partial T}, \qquad T = \frac{\partial E}{\partial S}$$

The entropy S and temperature T are conjugate variables.

Grandcanonical ensemble (chemical potential μ fixed)

The grandcanonical **thermodynamic potential** $\Omega=-pV$ as a function of two variables T and μ is related to the internal energy $E=\rho V$ via two-dimensional Legendre transformation

$$\Omega(T,\mu) = E(S,N) - TS - \mu N$$

By dividing by V, this may be expressed locally as

$$p(T,\mu) = Ts + \mu n - \rho(s,n) \qquad (1.2)$$

together with the conditions

$$s = \frac{\partial p}{\partial T}, \qquad n = \frac{\partial p}{\partial \mu}, \qquad T = \frac{\partial \rho}{\partial s}, \qquad \mu = \frac{\partial \rho}{\partial n} \qquad (1.3)$$

The entropy $S=sV$ and particle number $N=nV$ are conjugate to the temperature T and chemical potential μ, respectively

Introducing the specific **enthalpy** (or the specific **heat content**)

$$w = \frac{p + \rho}{n} \tag{1.4}$$

two useful thermodynamic identities follow

$$dw = Td\,\frac{s}{n} + \frac{1}{n}\,dp \tag{1.5}$$

TDS equation

$$d\,\frac{p}{T} = nd\,\frac{\mu}{T} - \rho d\,\frac{1}{T} \tag{1.6}$$

Gibbs-Duhem relation

Exercise No 2: Derive (1.5) and (1.6) from (1.2)-(1.4)

Basic Fluid Mechanics

Assume that matter is a **perfect fluid**
described by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \qquad (1.7)$$

u_μ – fluid velocity

p – pressure

ρ – energy density

From the energy momentum conservation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1.8}$$

using $u_\mu T^{\mu\nu}{}_{;\nu} = 0$, we obtain the **continuity equation**

$$\dot{\rho} + u^{\nu}{}_{;\nu}(\rho + p) = 0. \tag{1.9}$$

Combining this and (8) we obtain the **Euler equation**

$$(p + \rho)\dot{u}_\mu - p_{,\mu} + \dot{p} u_\mu = 0 \tag{1.10}$$

where

$$\dot{\rho} = u^{\nu}\rho_{,\nu} \quad \dot{p} = u^{\nu}p_{,\nu} \qquad \dot{u}^{\mu} = u^{\nu}u^{\mu}{}_{;\nu}$$

The covariant divergence of the fluid velocity $u^{\nu}{}_{;\nu}$ or the fluid **expansion rate** is, in the cosmological context, related to the **Hubble expansion rate** as $u^{\nu}{}_{;\nu} = 3H$.

Exercise No 3: Derive (1.9) and (1.10) from (1.7) and (1.8)

Isentropic and adiabatic fluid

A flow is said to be ***isentropic*** when the specific entropy *s/n* is constant, i.e., when

$$(s/n)_{,\mu} = 0 \qquad (1.11)$$

and is said to be ***adiabatic*** when *s/n* is constant along the flow lines, i.e., when

$$u^\mu (s/n)_{,\mu} = 0$$

As a consequence of (1.11) and the thermodynamic identity (1.5) (TdS equation) the Euler equation (1.10) simplifies to

$$u^\mu \left[(w u_\mu)_{;\nu} - (w u_\nu)_{;\mu} \right] = 0 \qquad (1.12)$$

In this case, we may introduce a scalar function φ such that

$$w u_\mu = \varphi_{,\mu} \qquad (1.13)$$

which obviously solves (1.12). This is called the ***potential flow.***

Exercise No 4: Derive (1.12) from (1.10) using (1.5) and (1.11)

Speed of sound

The propagation of perturbations in a fluid is assumed to be an isentropic process. The perturbations propagate at the speed of sound defined by

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_{s/n}$$

or equivalently

$$c_s^2 = \frac{n}{w}\left.\frac{\partial w}{\partial n}\right|_{s/n}$$

Here, the subscript s/n or denotes that the derivative is taken at constant specific entropy

Exercise No 4a: Show this equivalence using (1.4) and (1.5)

Lagrangian and Hamiltonian

The dream of all physicists is a comprehensive fundamental theory, which is often in popular scientific literature called the ‘’**theory of everything”**. Of course, nobody expects that this theory provides answers to all the issues, for example, the cause of a cancer, how the mind works, and so on. From the **theory of everything** we only require to explain basic processes in nature. Today most physicists share the following view of the world: the laws of nature are unambiguously described by the principle of some unique **action** (or Lagrangian) that fully defines the vacuum, the spectrum of elementary particles, forces and symmetries.

Classical description

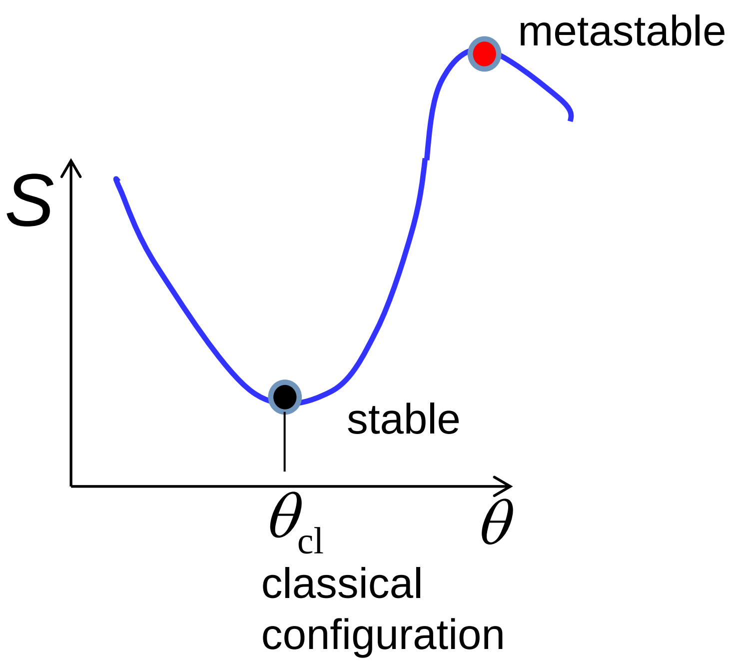
Consider a single self-interacting scalar field θ with action

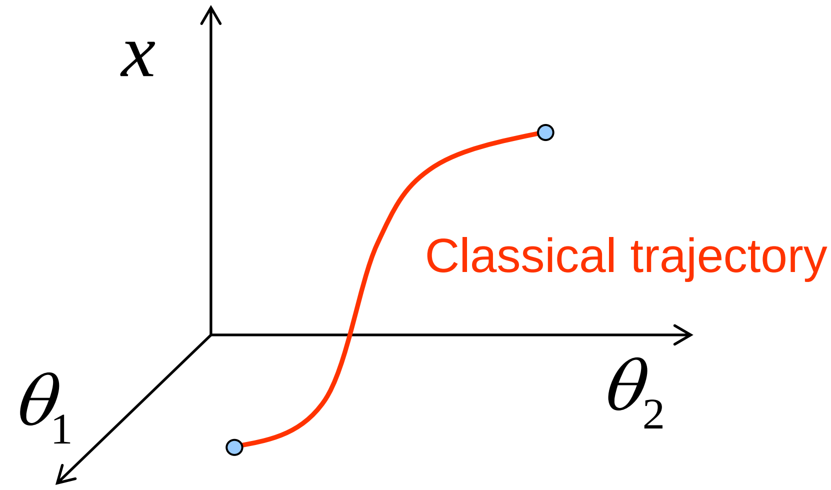
$$S = \int d^4x \sqrt{-\det g}\, \mathcal{L}(X, \theta)$$

where $X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$

One can generalize this to more fields $\theta_1, \theta_2, \cdots$ with $X_i = g^{\mu\nu}\theta_{i,\mu}\theta_{i,\nu}$

The principle of least action $\delta S = 0$ yields classical equations of motion





The Euler-Lagrange equations

From the requirement $\delta S = 0$ one finds the **classical** equations of motion

$$\left(\frac{\partial \mathcal{L}}{\partial \theta_{i,\mu}}\right)_{;\mu} = \frac{\partial \mathcal{L}}{\partial \theta_i} \qquad (1.14)$$

To each scalar field θ_i one can associate a current

$$J_{(i)\mu} = \mathcal{L}_{X_i} \theta_{i,\mu} \qquad \text{where} \qquad \mathcal{L}_X \equiv \frac{\partial \mathcal{L}}{\partial X}$$

If $\mathcal{L}$ does not depend on θ_i, the associated current is conserved, i.e.

$$J^{\mu}_{(i);\mu} = 0$$

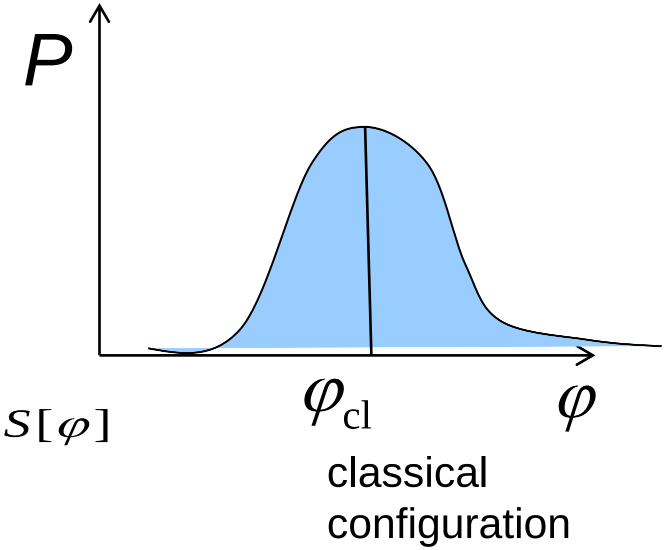
which follows directly from (14). This current conservation law is related to the shift symmetry

$$\theta_i \to \theta_i + c$$

Quantum description

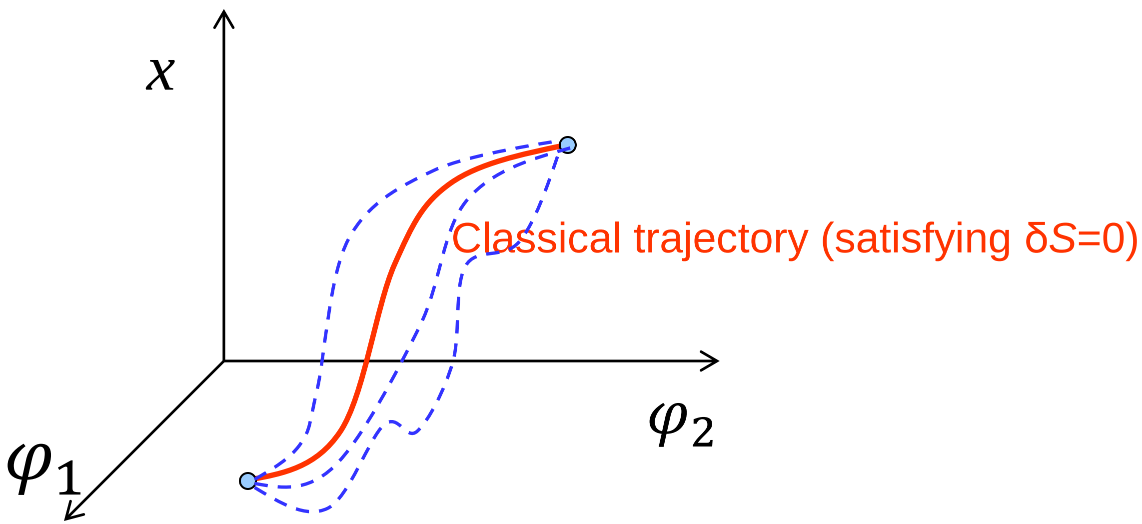
Through the action S we can define a probability distribution

$$P[\varphi_i] = \frac{e^{-S[\varphi_i]}}{Z}$$



Z = Partition function – path integral

$$Z = \sum_i e^{-S[\varphi_i]} \Rightarrow \int [d\varphi] e^{-S[\varphi]}$$



Field theoretical description of a fluid – *k*-essence

A field theory in which the Lagrangian is a function of a single field θ and its kinetic term $X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$, with action

$$S = \int d^4x \sqrt{-\det g}\,\mathcal{L}(X,\theta)$$

is called the ***k*-essence**. Assuming $X>0$ we define the stress tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}} = 2\mathcal{L}_X \theta_{,\mu}\theta_{,\nu} - \mathcal{L} g_{\mu\nu}, \qquad \text{where} \qquad \mathcal{L}_X \equiv \frac{\partial \mathcal{L}}{\partial X}$$

This may be written in a perfect fluid form as in equation (7),

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}$$

where we identify the velocity, pressure and energy density

$$u_\mu = \frac{\theta_{,\mu}}{\sqrt{X}} \qquad p = \mathcal{L} \qquad \rho = 2X\mathcal{L}_X - \mathcal{L}$$

The speed of sound can be expressed as

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_\theta = \frac{p_X}{\rho_X} = \frac{p_X}{p_X + 2Xp_{XX}} = \frac{p+\rho}{2X\rho_X} \qquad (14a)$$

Using the fluid description, the Euler-Lagrange equation

$$\left(\frac{\partial \mathcal{L}}{\partial \theta_{,\mu}}\right)_{;\mu} = \frac{\partial \mathcal{L}}{\partial \theta}$$

can be written in the form

$$u^{\mu}\left(\frac{\partial \mathcal{L}}{\partial \sqrt{X}}\right)_{;\mu} + u^{\nu}{}_{;\nu}\frac{\partial \mathcal{L}}{\partial \sqrt{X}} = \frac{\partial \mathcal{L}}{\partial \theta}$$ where $$u_{\mu} = \frac{\theta_{,\mu}}{\sqrt{X}}$$

This form is particularly convenient in the cosmological context. In FRW spacetime, θ depends on time only so $X = \dot{\vartheta}^2$, $u^{\mu} = (1,0,0,0)$ and

$$u^{\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}) = 3\frac{\dot{a}}{a} = 3H$$

The Euler-Lagrange equation turns into a 2nd order differential equation

$$\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} + 3H\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

Hamiltonian

Suppose for definiteness that $\mathcal{L}$ depends on a single field θ and its derivative $\theta_{,\mu}$. The canonical **Hamiltonian** is defined through a Legendre transformation which involves one pair of conjugate variables $\theta_{,0}$ and π^0

$$\mathcal{H}_{\text{can}}(\pi^0, \theta_{,i}, \theta) = \pi^0 \theta_{,0} - \mathcal{L}(\theta_{,0}, \theta_{,i}, \theta), \quad i = 1,2,3$$

where

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \theta_{,0}} \qquad \theta_{,0} = \frac{\partial \mathcal{H}_{\text{can}}}{\partial \pi^0}$$

The covariant Hamiltonian (de Donder-Weyl Hamiltonian)

$$\mathcal{H}(\pi^\mu, \theta) = \pi^\nu \theta_{,\nu} - \mathcal{L}(\theta_{,\mu}, \theta) \tag{1.15}$$

where

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \theta_{,\mu}} \qquad \theta_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu} \tag{1.16}$$

Historically, the covariant Hamiltonian was first introduced by De Donder 1930 and Weyl 1935 in the so called polysymplectic formalizm

Th. De Donder, *Th´eorie Invariantive Du Calcul des Variations,* Gaultier-Villars & Cia., Paris, France (1930).

H. Weyl, Annals of Mathematics 36, 607 (1935)

Recent references

J. Struckmeier, A. Redelbach, *Covariant Hamiltonian Field Theory*, Int. J. Mod. Phys. E 17, 435 (2008), arXiv:0811.0508

C. Cremaschini and M.Tessarotto, ``Manifest Covariant Hamiltonian Theory of General Relativity,'' Appl. Phys. Res. 8, 60 (2016), arXiv:1609.04422

k-essence Hamiltonian

In this case the Lagrangian $\mathcal{L}$ depends on the field θ and its kinetic term $X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$. The conjugate momentum field is given by

$$\pi_\theta^\mu = \frac{\partial \mathcal{L}}{\partial \theta_{,\mu}} = 2\mathcal{L}_X g^{\mu\nu}\theta_{,\nu}$$

For a timelike $\theta_{,\mu}$, i.e., for $X>0$, we may also define

$$\pi_\theta = \frac{\partial \mathcal{L}}{\partial \sqrt{X}} = \sqrt{g_{\mu\nu}\pi_\theta^\mu \pi_\theta^\nu}$$

Then, $\mathcal{H}$ is related to $\mathcal{L}$ via a one-dimensional Legendre transformation

$$\mathcal{H}(\theta, \pi_\theta) = \pi_\theta \sqrt{X} - \mathcal{L}(\theta, \sqrt{X})$$

with $$\pi_\theta = \frac{\partial \mathcal{L}}{\partial \sqrt{X}}$$ and $$\sqrt{X} = \frac{\partial \mathcal{H}}{\partial \pi_\theta}$$

Examples:

1) Canonical scalar field theory

$$\mathcal{L} = \frac{1}{2}X - V(\theta)$$

$$\mathcal{H} = \frac{1}{2}\pi_\theta^2 + V(\theta)$$

3) Scalar Born-Infeld field theory (tachyon condensate)

$$\mathcal{L} = -V(\theta)\sqrt{1 - X}$$

$$\mathcal{H} = \sqrt{V(\theta)^2 + \pi_\theta^2}$$

Hamilton field equations

The energy-momentum tensor corresponding to the k-essence Lagrangian

$$T_{\mu\nu} = 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \mathcal{L}g_{\mu\nu} = 2\mathcal{L}_X\,\pi_{\theta\mu}\pi_{\theta\nu} - g_{\mu\nu}\mathcal{L}$$

may be expressed as a perfect fluid (equation (1.7))

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \qquad (1.17)$$

with

$$p = \mathcal{L} \qquad \rho = 2X\mathcal{L}_X - \mathcal{L}$$

$$u_\mu = \frac{\pi_{\theta\mu}}{\pi_\theta} \qquad \pi_\theta = \sqrt{g_{\mu\nu}\pi_\theta^\mu\pi_\theta^\nu}$$

$\mathcal{H}$ is related to $\mathcal{L}$ through the Legendre transformation (1.15)

$$\mathcal{H}(\theta, \pi_\theta^\mu) = \pi_\theta^\nu \theta_{,\nu} - \mathcal{L}(\theta, \theta_{,\mu}) \qquad (1.15a)$$

It may be easily verified that this Hamiltonian is equal to the energy density

$$\rho = T^\mu{}_\mu + 3\mathcal{L} = \mathcal{H} \qquad (1.18)$$

The field variables are constrained by

$$\theta_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi_\theta^\mu}$$
$$\pi_\theta^\mu = \frac{\partial \mathcal{L}}{\partial \theta_{,\mu}} \qquad (1.19)$$

Exercise No 5: Prove $\rho = \mathcal{H}$ using (1.15a), (1.17), and (1.18)

Now we multiply the first equation by u^μ, take a covariant divergence of the second equation, use the Euler-Lagrange equation and the obvious relation

$$\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{\partial \mathcal{L}}{\partial \theta}$$

We obtain a set of two 1st order Hamilton's diff. equations

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial \pi_\theta} \qquad \dot{\pi}_\theta + 3H\pi_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} \qquad (1.20)$$

where

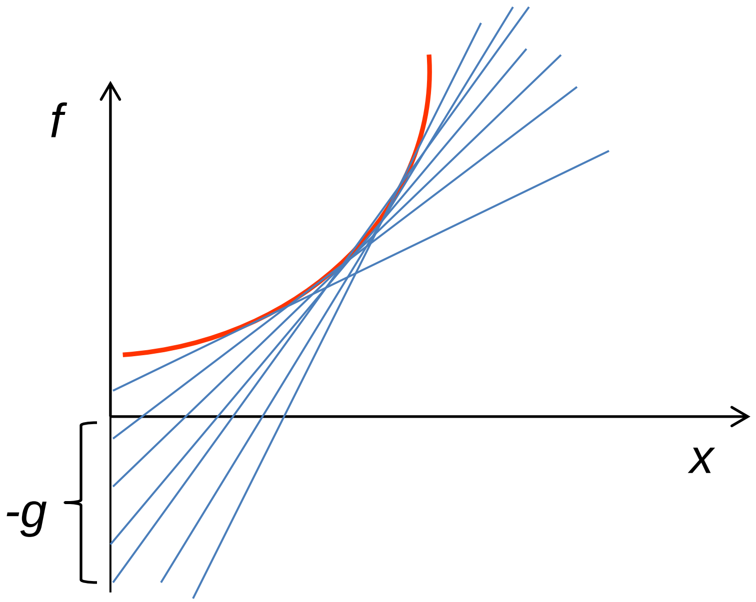
$$\dot{\theta} \equiv u^\mu \, \theta_{,\mu} \, , \qquad \dot{\pi}_\theta = u^\mu \, (\pi_\theta)_{,\mu}$$

and

$3H = u^\mu{}_{;\mu}$ is the fluid expansion rate

Exercise No 6: Derive eqs. (1.20) from (1.15a) and (1.19)

Thank you



By variing x_0 the function $f(x)$ may be regarded as the envelope of tangents

g is an implicit function of u

The generalization to n dimensions is straightforward:

$$f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} u_i x_i - g(u_1, u_2, ..., u_n)$$

with

$$u_i = \frac{\partial f}{\partial x_i}, \qquad x_i = \frac{\partial g}{\partial u_i}$$

Example: Tachyon condensate

The tachyon condensate is an effective Born-Infeld type Lagrangian

$$\mathcal{L} = -V(\theta)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

The tachyon field θ describes unstable modes in string theory

A. Sen, JHEP **0204** (2002); **0207** (2002).

A typical potential has minima at $\theta = \pm\infty$. Of particular interest are the inverse power law potential $V \propto \theta^{-n}$ and exponential potential $V \propto e^{-\omega\theta}$

Density of Matter in Space

The best agreement with cosmologic observations are obtained by the models with a flat space

According to Einstein's theory, a flat space universe requires critical matter density ρ_{cr} today

$\rho_{cr} \approx 10^{-29}$ g/cm^3

$\Omega = \rho / \rho_{cr}$ ratio of the actual to the critical density

For a flat space $\Omega=1$

What does the Universe consist of?

From astronomical observations:

luminous matter (stars, galaxies, gas ...) $\rho_{lum}/\rho_{cr} \leq$ **0.5%**

From the light element abundances and comparison with the Big Bang nucleosynthesis:

baryonic matter(protons, neutrons, nuclei) $\rho_{Bar}/\rho_{cr} \leq$ **5%**

Total matter density fraction $\Omega_M = \rho_M/\rho_{cr} \approx$ **0.31**

Accelerated expansion and comparison of the standard Big Bang model with observations requires that the **dark energy** density (vacuum energy) today $\Omega_\Lambda = \rho_\Lambda/\rho_{cr}$ **= 0. 69%**

Density fractions of various kinds of matter today with respect to the total density

$$\Omega_B = \frac{\rho_B}{\rho_{\text{tot}}} \approx 0.05 \quad \Omega_{\text{DM}} = \frac{\rho_{\text{DM}}}{\rho_{\text{tot}}} \approx 0.26 \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\text{tot}}} \approx 0.69$$

These fractions change with time but for a <u>spatially flat</u> Universe the following always holds:

$$\rho_{\text{tot}} = \rho_{\text{crit}}$$

Age of the Universe

Easy to calculate using the present observed fractions of matter, radiation and vacuum energy.
For a spatially flat Universe from the first Friedmann equation and energy conservation we have

$$H(a) = H_0(\Omega_\Lambda + \Omega_M a^{-3} + \Omega_R a^{-4})^{1/2}$$

$$H_0 = h \times 100\text{km s}^{-1}\text{Mpc}^{-1} = (14.5942\text{Gyr})^{-1}, \quad h = 0.67$$

The age of the Universe *T* can be calculated using

$$T = \int_0^T dt = \int_0^1 \frac{da}{aH}$$

Exercise No 9: Calculate *T* using Ω_Λ=0.69, Ω_M=0.31, Ω_R=0.

Hot DM refers to low-mass neutral particles that are still relativistic when galaxy-size masses ($\sim 10^{12} M_{\odot}$) are first encompassed within the horizon. Hence, fluctuations on galaxy scales are wiped out. Standard examples of hot DM are neutrinos and majorons. They are still in thermal equilibrium after the **QCD deconfinement** transition, which took place at $T_{\mathrm{QCD}} \approx 150$ MeV.
The cosmological-fluid pressure is not negligible. The equation of state similar to radiation

$$p_{\mathrm{DM}} \cong \rho_{\mathrm{DM}}/3$$

Hot DM particles have a cosmological number density comparable with that of microwave background photons, which implies an upper bound to their mass of a few tens of eV.

Warm DM particles are just becoming nonrelativistic when galaxy-size masses enter the horizon. Warm DM particles interact much more weakly than neutrinos. They decouple (i.e., their mean free path first exceeds the horizon size) at T>>$T_{\rm QCD}$. As a consequence, their mass is expected to be roughly an order of magnitude larger, than hot DM particles. The cosmological-fluid pressure is not negligible but small

$$p_{\rm DM} \ll \rho_{\rm DM}$$

Examples of warm DM are keV sterile neutrino, axino, or gravitino in soft supersymmetry breaking scenarios.

Cold DM particles are already nonrelativistic when even globular cluster masses ($\sim 10^6 M_\odot$) enter the horizon. Hence, their free path is of no cosmological importance. In other words, all cosmologically relevant fluctuations survive in a universe dominated by cold DM. The cosmological-fluid pressure is negligible. The equation of state is similar to dust

$$p \approx 0$$

The two main particle candidates for cold dark matter are the lowest supersymmetric *weakly interacting massive particles* (WIMPs) and the *axion*.

Time dependence of the DE density

Another important property of DE is that its density does not vary with time or **very weakly** depends on time. In contrast , the density of ordinary matter varies rapidly because of a rapid volume expansion.

The rough picture is that in the early Universe when the density of matter exceeded the density of DE the Universe expansion was slowing down. In the course of evolution the matter density decreases and when the DE density began to dominate, the Universe began to accelerate.

Dark Energy

Because gravity acts as an attractive force between astrophysical objects we expect that the expansion of the Universe will slowly decelerate.

However, recent observations indicate that the Universe expansion began to accelerate since about 5 billion years ago.

Repulsive gravity?

Accelerated expansion $\Rightarrow \Lambda \neq 0$

One possible explanation is the existence of a fluid with negative pressure such that

$$p + 3\rho < 0$$

and in the second Friedmann equation the universe acceleration $\ddot{a}$ becomes positive

cosmological constant $\Lambda =$ vacuum energy density with equation of state *p=-ρ.* Its negative pressure may be responsible for accelerated expansion!

New term: Dark Energy – fluid with negative pressure - generalization of the concept of vacuum energy

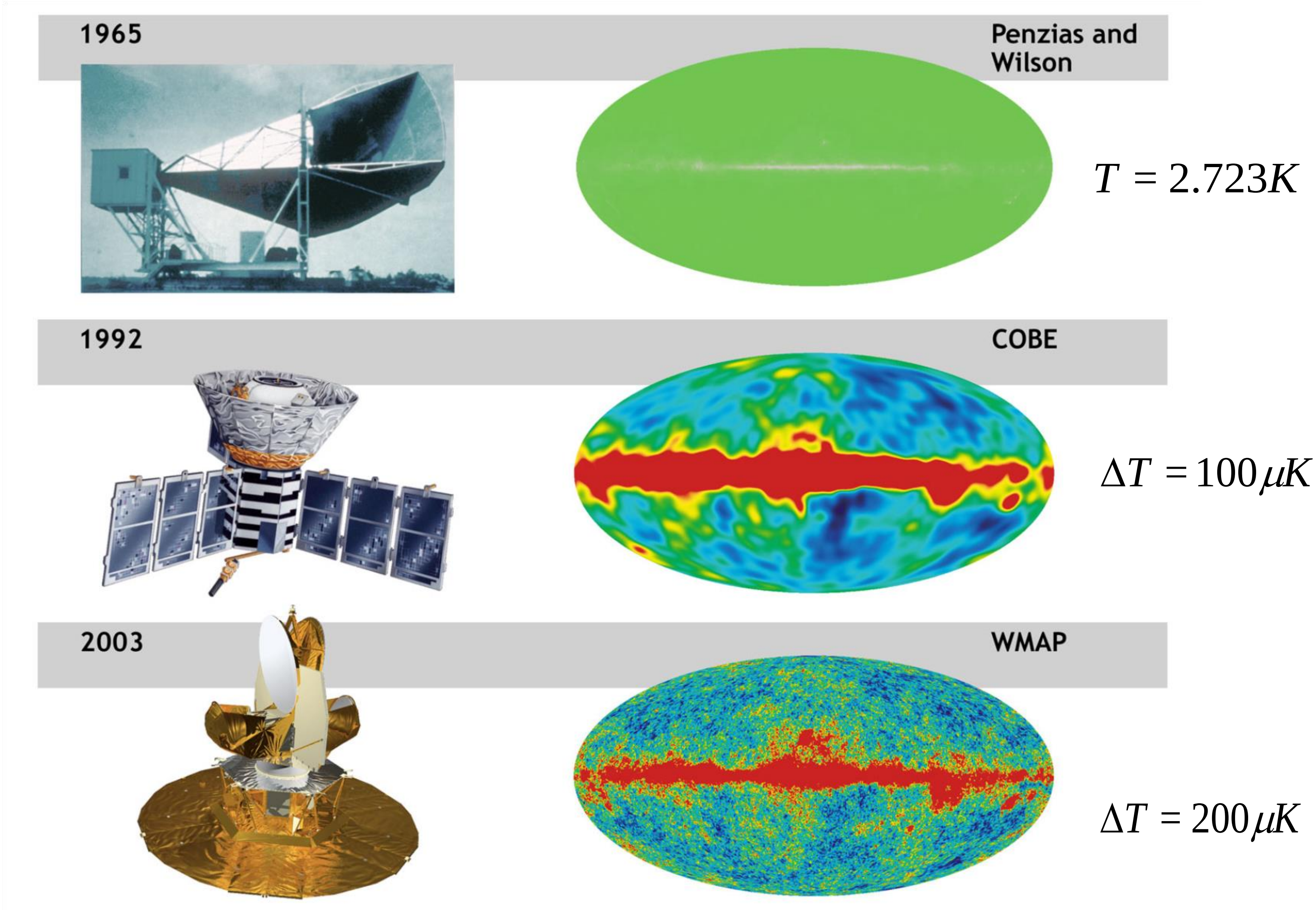
Problems with Λ

1) Fine tuning problem. The calculation of the vacuum energy density in field theory of the Standard Model of particle physics gives the value about 10^{120} times higher than the value of Λ obtained from observations. One possible way out is fine tuning: a rather unnatural assumption that all interactions of the standard model of particle physics somehow conspire to yield cancellation between various large contributions to the vacuum energy resulting in a small value of Λ , in agreement with observations

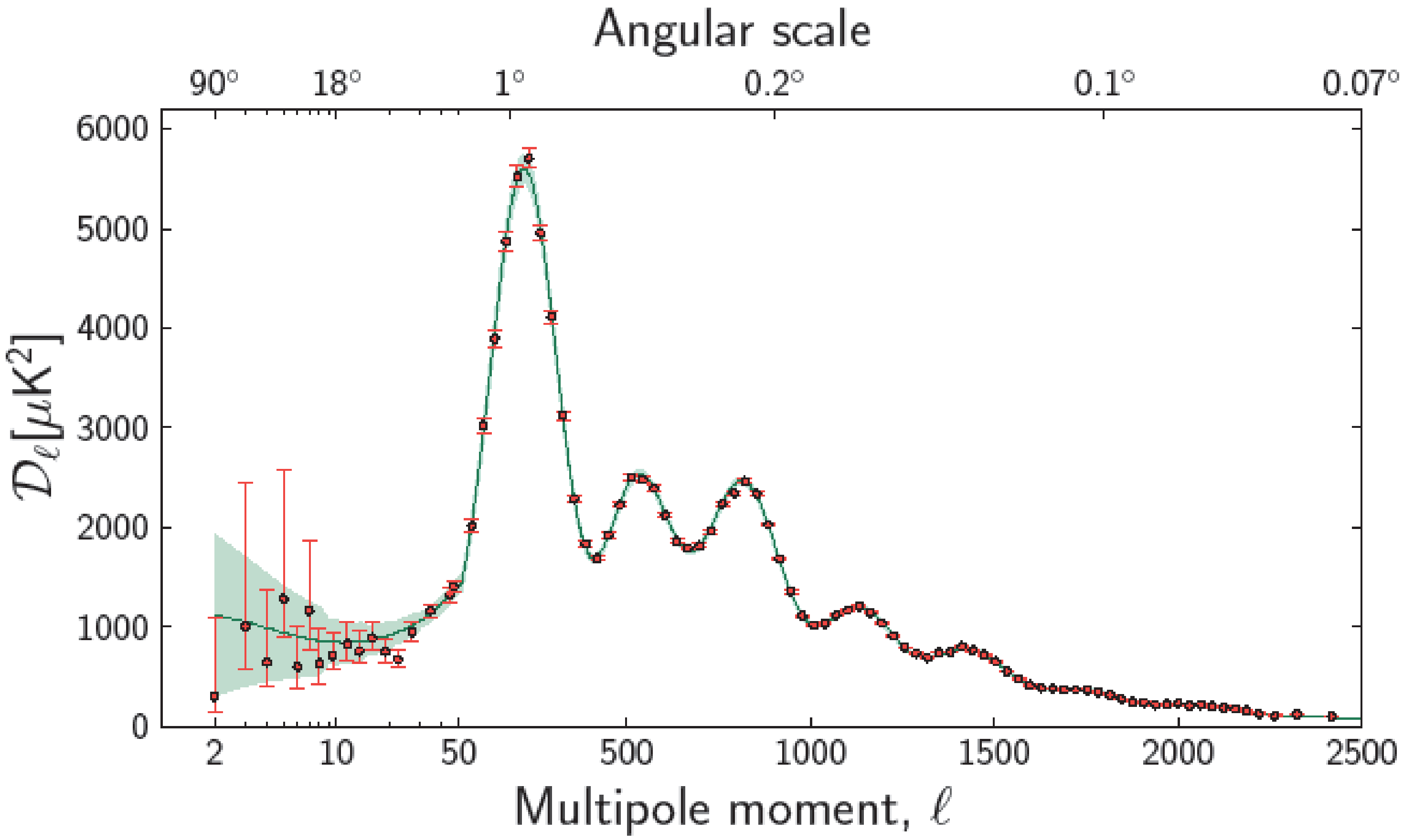
2) Coincidence problem. Why is this *fine tuned* value of Λ such that DM and DE are comparable today, leaving one to rely on anthropic arguments?

Most popular models of dark energy

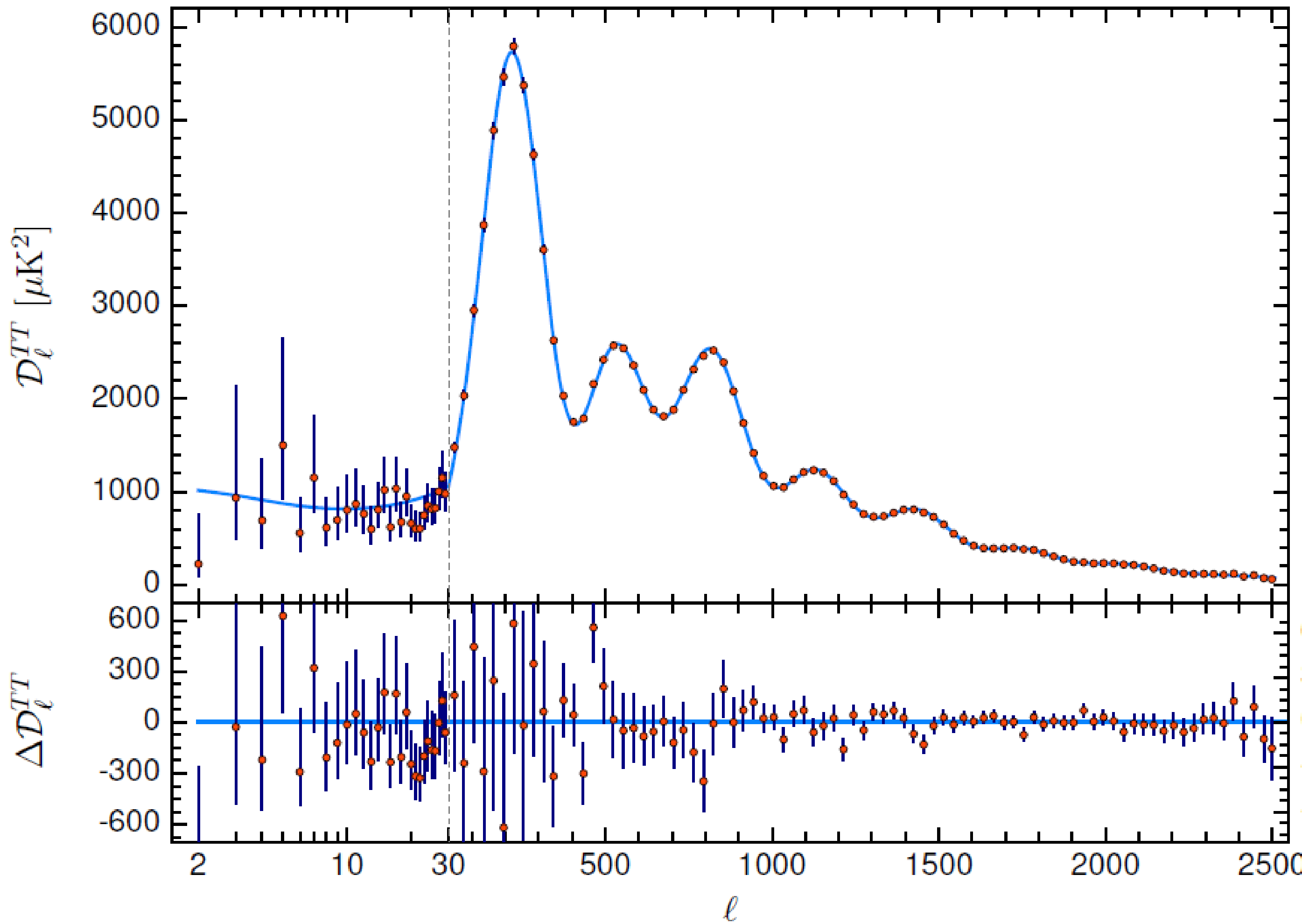
- Cosmological constant – vacuum energy density. Energy density does not vary with time.
- Quintessence – a scalar field with a canonical kinetic term. Energy density varies with time.
- Phantom quintessence – a scalar field with a negative kinetic term. Energy density varies with time.
- k-essence – a scalar field whose Lagrangian is a general function of kinetic energy. Energy density varies with time.
- Quartessence – a model of unifying of DE and DM. Special subclass of k-essence. One of the popular models is the so-called *Chaplygin gas*



Measuring CMB; the temperature map of the sky.



Angular (multipole) spectrum of the fluctuations of the CMB (Planck 2013)



6000
5000
4000
3000
2000
1000
0
$\mathcal{D}_\ell^{TT}$ [μK^2]
600
300
0
-300
-600
$\Delta\mathcal{D}_\ell^{TT}$
60
30
0
-30
-60
2
10
30
500
1000
1500
2000
2500
ℓ

Example: Tachyon condensate as CDM

The tachyon condensate is an effective Born-Infeld type Lagrangian

$$\mathcal{L} = -V(\theta)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

The tachyon field θ describes unstable modes in string theory

A. Sen, JHEP **0204** (2002); **0207** (2002).

A typical potential has minima at $\theta = \pm\infty$. Of particular interest is the inverse power law potential $V \propto \theta^{-n}$.

For $n > 2$, as the tachyon rolls near minimum, the pressure $p \equiv \mathcal{L} \to 0$ very quickly and one thus apparently gets pressure-less matter (dust) or cold dark matter.

L.R. Abramo and F. Finelli, PLB **575**(2002).

NB1: The solution (13) $wu_\mu = \varphi_{,\mu}$ is the relativistic analogue of a potential flow in nonrelativistic fluid dynamics

L.D. Landau, E.M. Lifshitz, Fluid Mechanics, Pergamon, Oxford,1993.

NB2: The potential flow (13) implies the isentropic Euler equation (12) but not the other way round. However if the fluid is ***isentropic*** and ***irrotational***, then equations (12) and (13) are equivalent. The fluid is said to be ***irrotational*** if its vorticity vanishes. The vorticity is defined as

$$\omega_{\mu\nu} = h^\rho_\mu h^\sigma_\nu u_{[\rho;\sigma]} \quad \text{where} \quad h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu$$

Vanishing vorticity, i.e., $\omega_{\mu\nu}=0$, implies

$$(wu_\mu)_{;\nu} - (wu_\nu)_{;\mu} = 0$$

A.H. Taub, Relativistic Fluid Mechanics, Ann. Rev. Fluid Mech. 10 (1978) 301

This equation is satisfied if and only if $wu_\mu = \varphi_{,\mu}$. Then, the Euler equation (12) is satisfied *identically.*

Hypersurfaces

In a *d*-dimensional spacetime manifold (bulk), a hypersurface is a p-dimensional (p<d) submanifold (subspace) which can be either spacelike, timelike, or null.

X^a – coordinates in the bulk

a,b=0,1,2,3, …, d-1

x^μ – coordinates on the hypersurface μ,ν=1,2,3, …, p

In particular, we will consider submanifolds of dimension p=d-1. For example, in the braneworld scenario, the Universe is a 3+1-dimensional timelike hypersurface in a 4+1-dimensional bulk

Defining equations

A d-dimensional hypersurface Σ in a $d+1$-dimensional bulk with metric G_{ab} can be selected either by

1) putting a restriction on the coordinates

$$\Phi(X^a) = \mathrm{const}$$

so that $d\Phi=0$ along the hypersurface, or by

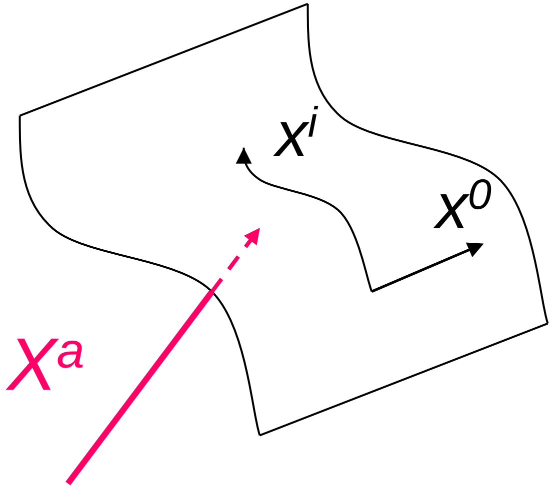
2) parametric equations of the form

$$X^a = X^a(x^\mu)$$

X^a – coordinates in the bulk

$a,b=0,1,2,3,\ldots,d$

x^μ – coordinates on the hypersurface $\mu,\nu=0,1,2,3,\ldots,d-1$



Normal vector

Because $d\Phi \equiv \Phi_{,a} dX^a = 0$, the vector $\Phi_{,a}$ is orthogonal to the displacements dX^a along Σ. Depending on the sign of the norm of the vector $\Phi_{,a}$ we have

$G^{ab}\Phi_{,a}\Phi_{,b}$
- >0 - spacelike hypersurface
- <0 - timelike hypersurface
- =0 - null hypersurface

If $G^{ab}\Phi_{,a}\Phi_{,b} \neq 0$, $\Phi_{,a}$ can be normalized

$$n_a = \frac{\Phi_{,a}}{\sqrt{|G^{ab}\Phi_{,a}\Phi_{,b}|}}$$

so that

$n_a =$
- 1 for a spacelike hypersurface (n_a is a timelike vector)
- -1 for a timelike hypersurface (n_a is a spacelike vector)

Induced metric

For the displacements dX^a on Σ we have

$$ds^2_\Sigma = G_{ab}dX^a dX^b = G_{ab}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}dx^\mu dx^\nu = g^{\text{ind}}_{\mu\nu}\, dx^\mu dx^\nu$$

The 4-tensor

$$g^{\text{ind}}_{\mu\nu} = G_{ab}\frac{\partial X^a}{\partial x^\mu}\frac{\partial X^b}{\partial x^\nu}$$

is called the induced metric, or first fundamental form, of the hypersurface Σ. This equation can be regarded as a coordinate transformation

$$G'_{cd} = G_{ab}\frac{\partial X^a}{\partial x^c}\frac{\partial X^b}{\partial x^d}$$

from $\{X^a\}$ to $\{x^c\}$ coordinate frames and we restrict the tensor G'_{ab} to the coordinates $a,b=0,1,2,\ldots,d-1$ which we denote by Greek letters μ,ν. From now on we restrict attention to $d=4$.

Projector

The tensor

$$h_{ab} = G_{ab} - \epsilon n_a n_b \qquad a, b = 0,1,2,3,4$$

is the projection tensor onto Σ, where n_a is a unit vector normal to Σ with $\epsilon = 1$ (-1) for a timelike (spacelike) vector n_a . Clearly $n^a h_{ab} = 0$.
One can show that the 4-tensor $h_{\mu\nu}$, $\mu, \nu = 0,1,2,3,$ is also an induced metric on Σ which is related to $g_{\mu\nu}^{\mathrm{ind}}$ by a coordinate transformation.
To show this, make a coordinate transformation $X^a = X^a(\tilde{X}^b)$ such that the normal vector in the new coordinates takes the form $\tilde{n}^a = \delta_y^a$, where the coordinate y is such that the defining equation for the hypersurface Σ is $y = \mathrm{const}$. In this coordinate frame the projector onto Σ is

$$\tilde{h}_{\mu\nu} = \tilde{G}_{\mu\nu} = \tilde{g}_{\mu\nu}^{\mathrm{ind}}$$

where the bulk metric components in new coordinates are obtained as usual

$$\tilde{G}_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial \tilde{X}^\mu} \frac{\partial X^b}{\partial \tilde{X}^\nu}$$

Extrinsic curvature

The projection of the covariant derivative of n_d

$$K_{ab} = h_a^c h_b^d n_{d;c}$$

restricted to the coordinates μ,ν=0,1,2,3, i.e., the 4-tensor $K_{\mu\nu}$, is called the extrinsic curvature, or second fundamental form, of the hypersurface Σ. In special coordinates, with $n^a = \delta_y^a$, we find

$$K_{\mu\nu} = n_{\mu;\nu} = -\Gamma_{\mu\nu}^a n_a$$

The trace K of K_{ab}

$$K \equiv G^{ab} K_{ab} = \; h^{ab} K_{ab} = n_{;a}^a$$

Einstein-Hilbert action

In 4+1 dimensional spacetime with boundary at a 4 dim. hypersurface Σ , the vacuum Einstein equations can be derived from the action

$$S = \frac{1}{8\pi G_5} \int d^5x \sqrt{-G} \left(-\frac{R^{(5)}}{2} - \Lambda_5 \right) + S_{\mathrm{GH}}$$

The Gibbons-Hawking boundary term is

$$S_{\mathrm{GH}} = \frac{\epsilon}{8\pi G_5} \int_\Sigma d^4x \sqrt{-\det h}\,(K - K_0)$$

where $\epsilon = \pm 1$ for a timelike (spacelike) hypersurface Σ and K_0 is the trace of the extrinsic curvature of Σ embedded in flat spacetime. The GH term is necessary to cancel a generally nonvanishing contribution of the boundary in the variation of the action δS.

Then, the variation principle $\delta S=0$ yields the Einstein equations in vacuum

$$R^{(5)}_{ab} - \frac{1}{2}R^{(5)}G_{ab} = \Lambda_5 G_{ab}$$

and junction conditions

$$\left[\left[K^{\alpha}_{\beta} - K\delta^{\alpha}_{\beta}\right]\right] = 8\pi G_5 T^{\alpha}_{\beta}$$

where T^{α}_{β} is the energy momentum tensor for matter localized on the hypersurface Σ and $[[f]]$ denotes the discontinuity of a function $f(x)$ across Σ, i.e.,

$$\left[\left[f(x)\right]\right] = \lim_{\varepsilon\to 0}\left(f(x+\varepsilon) - f(x-\varepsilon)\right)$$

The junction conditions prescribe the appropriate boundary conditions across a singular hypersurface Σ supported by a localized energy momentum tensor T^{α}_{β} .

Relativistic particle action

PARTICLE is a 0+1-dimdimensional object the dynamics of which in *d*+1-dimensional bulk is described by the relativistic pointlike-particle action

$$S_{\rm part} = -\int \sqrt{ds^2} = -\int d\tau \sqrt{\gamma} = -\int d\tau \sqrt{1-\dot{x}^2} \quad ,$$

where

$$ds^2 = G_{ab} dX^a dX^b, \quad \gamma = G_{ab} \frac{\partial X^a}{\partial \tau} \frac{\partial X^b}{\partial \tau}, \qquad a, b = 0, \ldots, d$$

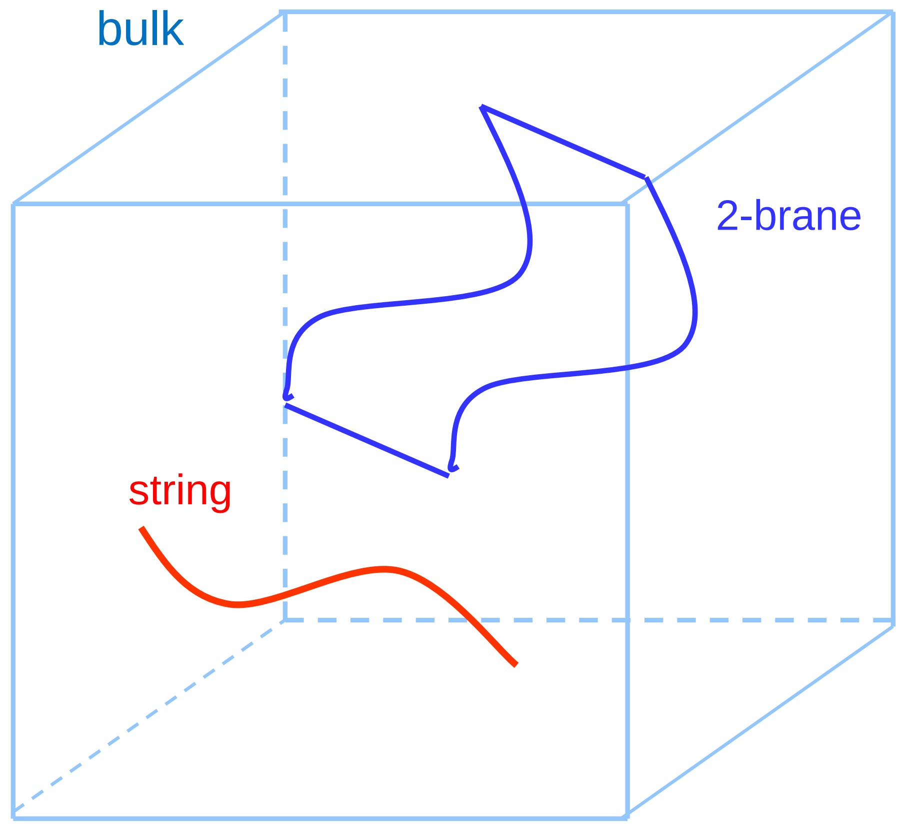
$$\dot{x}^2 = G_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} \qquad i, j = 1, \ldots, d \qquad \tau \equiv x^0$$

G_{ab} – metric in the bulk
X^a – coordinates in the bulk;
τ – synchronous time coordinate (G_{00}=1)

Strings and (mem)branes

STRING is a 1+1-dimdimensional object moving in the d+1 dimensional bulk

p-BRANE is a *p*+1-dim. object that generalizes the concept of membrane (2-brane) or string (1-brane)



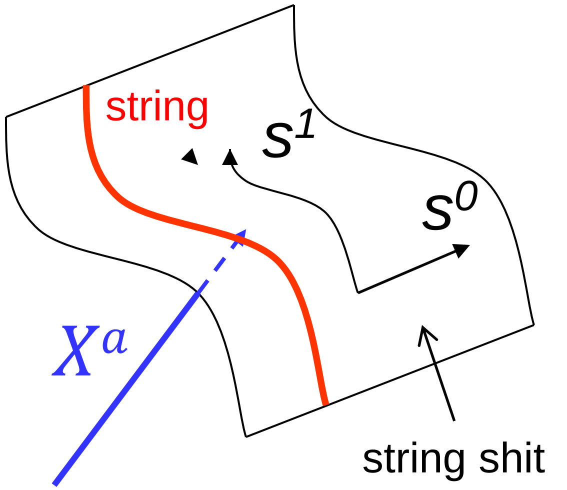
String action

The dynamics of a ***STRING*** in *d*+1-dimensional bulk is described by the Nambu-Goto action (generalization of the relativistic particle action)

$$S_{\text{string}} = -T\int d\tau d\sigma\sqrt{-\det(g^{\text{ind}}_{\alpha\beta})}$$

where $g_{\alpha\beta}$ is induced metric ("pull back")

$$g^{\text{ind}}_{\alpha\beta} = G_{ab}\frac{\partial X^a}{\partial s^\alpha}\frac{\partial X^b}{\partial s^\beta} \qquad \alpha,\beta = 0,1$$



T – string tension

X^a – coordinates in the bulk;

$s^0 \equiv \tau$ – timelike coordinate on the string sheet

$s^1 \equiv \sigma$ – spacelike coordinate on the string sheet

Brane action

The dynamics of a ***p-BRANE*** in d+1-dimensional bulk is described by the Nambu-Goto action similar to the string action. Nambu-Goto action for a 3-brane embedded in a 4+1 dim space-time (bulk)

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-\det h_{\mu\nu}}$$

σ – brane tension

$$h_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu}$$

– induced metric

G_{ab} – metric in the bulk

X^a – coordinates in the bulk

$a,b=0,1,2,3,4$

x^μ – coordinates on the brane

$\mu,\nu=0,1,2,3$