Introduction to Holography: Tutorials

R. C. Rashkov[a], K. Hristov[a] & T. Vetsov[a]

[a] Department of Physics, Sofia University, Bulgaria.

rash,khristov,vetsov@phys.uni-sofia.bg

CERN - SEENET-MTP - ICTP SCHOOL
Gravitation, Cosmology and Astroparticle Physics (BS2022)
4 – 10 September 2022, Belgrade, Serbia

Outline

- Extra dimensions and partition functions
- Scalar electrodynamics
- Classical conformal invariance
- Euclidean path integral in quantum mechanics
- Euclidean conformal group (Homework)
- AdS coordinates (Homework)
- Scalar field in AdS (Homework)
- Scalar field on a circle (Homework)
- Feynman rules (Homework)
- Scalar propagator (Homework)
- Gauge fixing for massive vector fields (Homework)

Problem 1.1. Square well in one dimension (a reminder)

Problem 1.1: Consider the time-independent Schrödinger equation

$$\Big[-\frac{\hbar^2}{2m}\partial^2_{xx}+V(x)-E\Big]\psi(x)=0. \qquad (1)$$

with the one-dimensional square-well potential of infinite height:

$$V(x)=\begin{cases} 0,\ \text{if}\, x\in(0,a), \quad a>0,\ \&\ \psi(0)=0=\psi(a), \\ \infty,\ \text{if}\, x\notin(0,a)\ \Rightarrow\ \psi(x)=0. \end{cases} \qquad (2)$$

Determine the wave function $\psi_k(x)$ and the energy spectrum E_k, where k is the principle quantum number.

Problem 1.2. Extra dimensions and partition functions

Problem 1.2 (B. Zwiebach): Add an extra circular dimension y with a small radius R to the square-well with $(x, y) = (x, y + 2\pi R)$. Now the particle moves on a cylinder with length a and circumference $2\pi R$. The potential $V(x, y)$ remains the same as in (2) and is y-independent.

a) Determine $\psi_{k,l}(x, y)$ and $E_{k,l}$. At what probing energies the effects of an extra dimension can be observed?

b) Determine the statistical partition function $\mathcal{Z}(a, R)$ and show that, at high temperatures ($\beta = (kT)^{-1} \to 0$), the effects of the extra dimension are visible. Evaluate $\mathcal{Z}$ in the regime $\frac{\hbar^2}{ma^2} \leq kT \leq \frac{\hbar^2}{mR^2}$ and include the leading correction due to the small extra dimension.

Hint: The Schrödinger equation in $2d$:

$$\left[-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x, y) - E\right]\psi(x, y) = 0. \tag{3}$$

Solution to Problem 1.1

In Sectors $x < 0$ and $x > a$ one has $V(x) = \infty$ and $\psi(x) = 0$.
In Sector $0 \le x \le a$ one has $\psi(0) = 0 = \psi(a)$ and $V(x) = 0$, hence (1):

$$\psi''(x) + \mu^2 \psi(x) = 0, \quad \mu^2 = \frac{2mE}{\hbar^2}. \tag{4}$$

The characteristic equation is $\lambda^2 + \mu^2 = 0$, hence $\lambda = \pm i|\mu|$ giving a solution

$$\psi(x) = C_1 e^{i|\mu|x} + C_2 e^{-i|\mu|x}. \tag{5}$$

Simplify using Euler's identity $e^{\pm i\varphi} = \cos(\varphi) \pm i \sin(\varphi)$:

$$\psi(x) = A \cos\big(|\mu|x\big) + B \sin\big(|\mu|x\big), \tag{6}$$

where $A = (C_1 + C_2) \in \mathbb{C}$ and $B = i(C_1 - C_2) \in \mathbb{C}$.

Solution to Problem 1.1

Impose boundary conditions:

$$0 = \psi(0) = A\cos 0 + B\sin 0 \quad \Rightarrow \quad A = 0, \tag{7}$$

$$0 = \psi(a) = B\sin\left(|\mu|a\right) \quad \Rightarrow \quad |\mu|a = k\pi, \quad k = 0, 1, 2, ..., \quad B \neq 0. \tag{8}$$

The spectrum follows from Eqs. (4) and (8):

$$E_k = \frac{\hbar^2}{2m}\left(\frac{k\pi}{a}\right)^2. \tag{9}$$

The wave function ($k = 0$ is not allowed since it would make the wave function vanish everywhere):

$$\psi_k(x) = B_k \sin\left(\frac{k\pi}{a}x\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{k\pi}{a}x\right), \quad k = 1, 2, ... \tag{10}$$

Normalization condition (the easy way):

$$1 = \langle\psi(x)|\psi(x)\rangle = \int_0^a |\psi(x)|^2 dx = B_k^2 \int_0^a \sin^2\left(\frac{k\pi x}{a}\right)dx \Rightarrow B_k = \sqrt{\frac{2}{a}}. \tag{11}$$

Normalization condition (the right way): $\delta_{mn} = \langle\psi_m(x)|\psi_n(x)\rangle$.

Solution to Problem 1.2a

Use separation of variables in (3), $\psi(x,y) = \psi(x)\phi(y)$:

$$-\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} - \frac{\hbar^2}{2m}\frac{\phi''(y)}{\phi(y)} = E, \quad x \in (0,a) \;\&\; y = y + 2\pi R. \tag{12}$$

The x-dependent and y-dependent terms must separately be constant ($-\mu^2$ and $-\nu^2$), thus the energy spectrum becomes:

$$\frac{\hbar^2}{2m}(\mu^2 + \nu^2) = E. \tag{13}$$

The solutions are given by

$$\psi(x) = a_1 \sin(\mu x) + a_2 \cos(\mu x), \tag{14}$$

$$\phi(y) = b_1 \sin(\nu y) + b_2 \cos(\nu y). \tag{15}$$

Imposing $\psi(0) = \psi(a)$ one has $a_2 = 0$ and $\mu a = k\pi$, hence

$$\psi_k(x) = a_k \sin\left(\frac{k\pi x}{a}\right), \quad k = 1, 2, ... \tag{16}$$

Solution to Problem 1.2a

We now impose $\phi(y) = \phi(y + 2\pi R)$:

$$b_1 \sin(\nu y) + b_2 \cos(\nu y) = b_1 \sin(\nu y + 2\pi R\nu) + b_2 \cos(\nu y + 2\pi R\nu). \qquad (17)$$

Expand by $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$:

$$\begin{aligned} &b_1 \big[\sin(\nu y)\big(1 - \cos(2\pi R\nu)\big) - \cos(\nu y) \sin(2\pi R\nu)\big] + \\ &b_2 \big[\cos(\nu y)\big(1 - \cos(2\pi R\nu)\big) + \sin(\nu y) \sin(2\pi R\nu)\big] = 0. \end{aligned} \qquad (18)$$

Use $1 - \cos(2\pi R\nu) = 2\sin^2(\pi R\nu)$ & $\sin(2\pi R\nu) = 2\sin(\pi R\nu)\cos(\pi R\nu)$. Obviously, a common factor of $\sin(\pi R\nu)$ is going to appear. Because we want to keep both sine and cosine terms in the wave function (i.e. $b_1 \neq 0$ and $b_2 \neq 0$), one has to impose $\sin(\pi R\nu) = 0$, thus $\pi R\nu = l\pi$:

$$\phi_l(y) = b_l \sin\left(\frac{l}{R} y\right) + c_l \cos\left(\frac{l}{R} y\right), \quad l = 0, 1, 2, ... \qquad (19)$$

$$E_{k,l} = \frac{\hbar^2}{2m}(\mu^2 + \nu^2) = \frac{\hbar^2}{2m}\left[\left(\frac{k\pi}{a}\right)^2 + \left(\frac{l}{R}\right)^2\right]. \text{ Interpret!} \qquad (20)$$

Solution to Problem 1.2b

The partition function is given by

$$\mathcal{Z}(a,R)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}e^{-\beta E_{k,l}}=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\exp\bigg\{-\frac{\beta\hbar^2}{2m}\bigg[\Big(\frac{k\pi}{a}\Big)^2+\Big(\frac{l}{R}\Big)^2\bigg]\bigg\}$$
$$=\sum_{k=1}^{\infty}\exp\bigg\{-\frac{\beta\hbar^2}{2m}\Big(\frac{k\pi}{a}\Big)^2\bigg\}\bigg(1+\sum_{l=1}^{\infty}\exp\bigg\{-\frac{\beta\hbar^2}{2m}\Big(\frac{l}{R}\Big)^2\bigg\}\bigg). \quad (21)$$

Hence, the partition function factorizes to

$$\mathcal{Z}(a,R)=Z(a)\big(1+Z(\pi R)\big), \quad (22)$$

where (r stands for a and also for πR):

$$Z(r)=\sum_{n=1}^{\infty}\exp\bigg\{-\frac{\beta\hbar^2}{2m}\Big(\frac{n\pi}{r}\Big)^2\bigg\}. \quad (23)$$

Note: The exact solution is given by a theta function $\theta_3(0,u)$:

$$Z(r)=\frac{1}{2}\vartheta_3\left(0,e^{-\frac{\pi^2\beta\hbar^2}{2mr^2}}\right)-\frac{1}{2},\quad \vartheta_3(u,q)=2\sum_{n=1}^{\infty}q^{n^2}\cos(2nu)+1. \quad (24)$$

An approximate solution to Problem 1.2b

$$Z(r) \approx \int_1^\infty dn \exp\left\{ -\frac{\beta\hbar^2}{2m}\left(\frac{n\pi}{r}\right)^2 \right\} = \int_1^\infty dn\, e^{-z^2 n^2}, \;\; z^2 = \frac{\beta\hbar^2}{2m}\left(\frac{\pi}{r}\right)^2. \quad (25)$$

Change variables to $z^2 n^2 = t^2$, i.e. $zn = t$, $zdn = dt$, and $t = z$ at $n = 1$:

$$Z(r) = \frac{1}{z}\int_z^\infty dt\, e^{-t^2} = \frac{2}{\sqrt{\pi} z}\mathrm{erfc}(z) = \frac{\sqrt{mr}}{\sqrt{2\pi\beta}\hbar}\mathrm{erfc}\left(\frac{\pi\sqrt{\beta}\hbar}{\sqrt{2mr}}\right), \quad (26)$$

where erfc(z) is the complementary error function (https://dlmf.nist.gov):

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2} dt, \quad (27)$$

Expansion around $\beta = 0$ (high temperature limit):

$$\mathcal{Z}(a, R) = Z(a)\big(1 + Z(\pi R)\big) = \frac{amR}{2\beta\hbar^2} - \frac{\sqrt{m\pi}R}{\sqrt{2\beta}\hbar} + O\big(\beta^{1/2}\big). \quad (28)$$

Therefore, at high temperature, one has a dominant effect of the extra dimension.

An approximate solution to Problem 1.2b

With an asymptotic expansion around $z = \infty$ of the erfc(z)

$$\mathrm{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m (1/2)_m z^{-1-2m}, \tag{29}$$

one finds the expansion around $\beta = \infty$ (low temperature limit):

$$\mathcal{Z}(a, R) = e^{-\frac{\pi^2 \beta \hbar^2}{2a^2 m}} \left(\frac{a^2 m^{3/2} R}{\sqrt{2}\pi^{3/2}\beta^{3/2}\hbar^3} - \frac{a^4 m^{5/2} R}{\sqrt{2}\pi^{7/2}\beta^{5/2}\hbar^5} + O\big(\beta^{-7/2}\big) \right). \tag{30}$$

Interpret the result?

Consider the medium regime for **homework**: $\frac{\hbar^2}{ma^2} \leq \frac{1}{\beta} \leq \frac{\hbar^2}{mR^2}$, i.e. $\frac{mR^2}{\hbar^2} \leq \beta \leq \frac{ma^2}{\hbar^2}$.

Problem 2: Scalar electrodynamics

Consider a complex scalar field ϕ in $D = d + 1$ dimensions coupled to a $U(1)$ gauge field, A_μ,

$$\mathcal{L}_{kin} = -(D_\mu \phi)^* D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \quad D_\mu := \partial_\mu - ieA_\mu . \tag{31}$$

a) Determine the mass dimensions of the fields and the coupling constant e - is the theory (classically) conformal invariant? Is the theory gauge invariant?
b) Write the equations of motion and the Feynman rules for the resulting theory (known as scalar electrodynamics).
c) Now add the term

$$\mathcal{L}_\lambda = \lambda_1 \phi^* \phi + \lambda_2 (\phi^* \phi)^2 \ , \tag{32}$$

where $\lambda_{1,2}$ are real coupling constants. Is there and what is the change in the equations of motion, Feynman rules and conformal/gauge invariance after the addition of $\mathcal{L}_\lambda$? Draw your favorite 1-loop Feynman diagram obeying the rules and write down its corresponding expression with the integral over the internal momentum.

Solutions to Problem 2a

Start with the mass dimension of the action, which is always zero in natural units ($\hbar = 1 = c$, i.e. $[J.s] = [ML^2T^{-1}] \to [MM^{-2}M] = [M^0] = 0$):

$$S = \int d^D x \mathcal{L} \quad \Rightarrow \quad [S] = [M^0] = 0, \quad [\mathcal{L}] = D. \tag{33}$$

$$[x] = [M^{-1}] = -1, \quad [\partial_x] = [M^1] = 1, \quad [d^D x] = [M^{-D}] = -D. \tag{34}$$

$$\begin{aligned} \int d^D x \, \mathcal{L}_{\phi,A} &= -\int d^D x (D_\mu \phi)^* D^\mu \phi \\ &= -\int d^D x (\partial_\mu \phi^* + ieA_\mu \phi^*)(\partial^\mu \phi - ieA^\mu \phi) \\ &= -\int d^D x (\partial_\mu \phi^* \partial^\mu \phi - ieA^\mu \phi \partial_\mu \phi^* + ieA_\mu \phi^* \partial^\mu \phi + e^2 A_\mu A^\mu \phi^* \phi). \end{aligned} \tag{35}$$

$$0 = \left[\int d^D x \partial_\mu \phi^* \partial^\mu \phi \right] = [M^{-D} M^1 M^a M^1 M^a] = [M^{-D+2+2a}] = -D+2+2a, \tag{36}$$

hence $a = (D-2)/2$, and the mass dimension of the scalar field is $[\phi] = a = (D-2)/2$. In $D = 4$: $[\phi] = 1$.

Solutions to Problem 2a

$$-\frac{1}{4}\int d^Dx F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^Dx(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{4}\int d^Dx(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu). \quad (37)$$

$$0 = \left[\int d^Dx\, \partial_\mu A_\nu \partial^\mu A^\nu\right] = [M^{-D}M^1 M^b M^1 M^b] = -D + 2b + 2,$$

hence $[A_\mu] = b = (D-2)/2 = a = [\phi]$.

$$0 = \left[\int d^Dx e A^\mu \phi \partial_\mu \phi^*\right] = [M^{-D}M^c M^b M^a M^1 M^a] = -D + c + 3a + 1,$$

$$0 = \left[\int d^Dx e^2 A_\mu A^\mu \phi^* \phi\right] = [M^{-D}M^{2c} M^b M^b M^a M^a] = -D + 2c + 4a.$$

Both equations lead to $c = (4-D)/2$, thus the mass dimension of the coupling is $[e] = (4-D)/2$. In $D=4$: $[e]=0$ is dimensionless and there is no scale, thus the theory is classically conformally invariant! in $D=4$.
Homework: test for local gauge invariance under ($\alpha(x) \in \mathbb{R}$):

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x), \quad \phi'(x) = e^{i\alpha(x)}\phi(x), \quad \phi'^*(x) = e^{-i\alpha(x)}\phi^*(x).$$

Solutions to Problem 2b and c (hints and homework)

2b. Hints:

$$\mathcal{L}_{kin} = -(D_\mu\phi)^* D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= -(\partial_\mu\phi^*\partial^\mu\phi - ieA^\mu\phi\partial_\mu\phi^* + ieA_\mu\phi^*\partial^\mu\phi + e^2A_\mu A^\mu\phi^*\phi)$$
$$- \frac{1}{4}(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu). \quad (38)$$

You can simplify the Lagrangian by $j^\mu = ie\big(\phi^*\partial^\mu\phi - (\partial^\mu\phi)^*\phi\big)$. Use Lagrange-Euler:

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}, \quad \frac{\partial\mathcal{L}}{\partial A_\beta} = \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}. \quad (39)$$

Don't forget to choose a gauge $\partial_\mu A^\mu = 0$.

2c. Hints:

$$\mathcal{L}_\lambda = \lambda_1\phi^*\phi + \lambda_2(\phi^*\phi)^2 \ , \quad (40)$$

$$D = [\lambda_1\phi^*\phi] = [M^f M^a M^a] = f + D - 2 \quad \Rightarrow \quad f = 2 \quad \Rightarrow \quad [\lambda_1] = 2. \quad (41)$$

$$D = [\lambda_2(\phi^*\phi)^2] = [M^g M^{2a} M^{2a}] = g + 2(D-2) \ \Rightarrow \ g = 4 - D \ \Rightarrow \ [\lambda_2] = 4 - D. \quad (42)$$

Therefore, the mass term $[\lambda_1] = 2 \neq 0$ breaks conformal invariance in any dimension, while in $D = 4$ the coupling $[\lambda_2] = 0$ is scale invariant.

Solutions to Problem 2b and c (hints and homework)

2c. Hints: You have three interacting fields: two scalar bosons ϕ and ϕ^* and a vector boson A_μ. Draw Feynman diagrams (in momentum space) using the internal and the external lines (propagators) for these fields. You can take advantage of the S-matrix time-ordered perturbation series and the Wick contraction theorem or the path integral formalism to derive the Feynman diagrams. Remember for every vertex to insert a factor of $-i\lambda_2$. Don't forget also to account for the conservation of the energy-momentum at every vertex: $\sim \delta(\sum p_{in} - \sum p_{out})$.

Problem 3: Classical conformal invariance

Consider the scalar field action in D-dimensions

$$\mathcal{S} = \int \mathrm{d}^D x \left(-\frac{1}{2}(\partial_\mu \phi)^2 - g\phi^3 - \lambda\phi^4 - k\phi^6 \right). \tag{43}$$

Determine which of the coupling constants $\{g, \lambda, k\}$ is allowed to be non-vanishing in a classically conformally invariant theory for different values of D. Which are the special values of D allowing for interacting classically conformal theories?

Problem 3: solution

Remember that $[\phi] = a = (D-2)/2$ and $[\mathcal{L}] = D$:

$$D = [g\phi^3] = [M^y M^{3(D-2)/2}] = y + \frac{3}{2}(D-2) \;\Rightarrow\; y = \frac{6-D}{2} = [g]. \quad (44)$$

$$D = [\lambda\phi^4] = [M^u M^{4(D-2)/2}] = u + 2(D-2) \;\Rightarrow\; u = 4 - D = [\lambda]. \quad (45)$$

$$D = [\lambda\phi^6] = [M^v M^{6(D-2)/2}] = v + 3(D-2) \;\Rightarrow\; v = -2(D-3) = [k]. \quad (46)$$

Interpretation:

- In $D = 6$: $g = 0$ and the ϕ^3 theory is classically conformal.
- In $D = 4$: $\lambda = 0$ and the ϕ^4 theory is classically conformal.
- In $D = 3$: $k = 0$ and the ϕ^6 theory is classically conformal.

Problem 4: Euclidean path integral in quantum mechanics

In quantum field theory, the Euclidean path integral on a compact time direction of period β has the interpretation of the statistical/thermal partition function Z_β if bosons are identified periodically and fermions anti-periodically. Here we change the boundary conditions and make all fields periodic (still, bosons are commuting and fermions anti-commuting), which corresponds to a different sort of observable partition function, a supersymmetric index Z_W also known as the Witten index.

Consider the double harmonic oscillator

$$H = \hbar\omega(a^\dagger a + b^\dagger b) = \bar{\omega}(n_b + n_f), \tag{47}$$

which can be interpreted as a coupled bosonic and fermionic oscillator with corresponding particle numbers $n_b \in \mathbb{Z}_+$ and $n_f \in \{0, 1\}$. The eigenstates of the Hamiltonian are therefore $|n_f, n_b\rangle$ with eigenvalues (energies) $\hbar\omega(n_b + n_f)$. Keeping in mind that the operators $a, a^\dagger$ are bosonic, while $b, b^\dagger$ are fermionic, evaluate the corresponding Euclidean path integral

$$Z_W = \int \frac{\mathrm{d}a^\dagger \mathrm{d}a}{2\pi i} \mathrm{d}b^\dagger \mathrm{d}b \; e^{-\beta H} \; . \tag{48}$$

Problem 4: Euclidean path integral in quantum mechanics

This path integral is a way of evaluating the Witten index and has the interpretation of counting the invariant quantity of bosonic minus fermionic states (a topological quantum number, which can be used to determine whether SUSY is exact or broken),

$$Z_W = Tr\big[(-1)^{n_f}\, e^{-\beta H}\big]. \tag{49}$$

Based on the explicit knowledge of the spectrum of the double harmonic oscillator, show that your result for Z_W indeed matches the expectation derived from the path integral. How does the final answer depend on β and why?

Problem 4: hints and solutions

Commutation/anticommutation relations:

$$[a, a^\dagger] = \hbar \hat{1}, \quad \{b, b^\dagger\} = \hbar \hat{1}, \quad [a, b] = 0, \quad b^2 = 0 = b^{\dagger 2}. \tag{50}$$

Fock space

$$a|n_f, n_b\rangle = |n_f, n_b - 1\rangle, \quad b|n_f, n_b\rangle = |n_f - 1, n_b\rangle, \quad n_f = 0, 1, \tag{51}$$

$$a^\dagger|n_f, n_b\rangle = |n_f, n_b + 1\rangle, \quad b^\dagger|n_f, n_b\rangle = |n_f + 1, n_b\rangle. \tag{52}$$

The ground state is non-degenerate, but there is a double degeneracy of all the excited levels:

Energy level	boson state	fermion state
0	$\|0, 0\rangle$	
$\hbar\omega$	$\|0, 1\rangle$	$\|1, 0\rangle$
$2\hbar\omega$	$\|0, 2\rangle$	$\|1, 1\rangle$
$3\hbar\omega$	$\|0, 3\rangle$	$\|1, 2\rangle$
$4\hbar\omega$	$\|0, 4\rangle$	$\|1, 3\rangle$

See D. Tong: http://www.damtp.cam.ac.uk/user/tong/susyqm.html.

Problem 4: hints and solutions

Witten index (if $Z_W = Z_W(\beta)$ one has a broken suppersymmetry):

$$Z_W = \mathrm{Tr}\big[(-1)^{n_f} e^{-\beta H}\big] = \sum_{n_b=0}^{\infty} \sum_{n_f=0}^{1} \langle n_b, n_f | (-1)^{n_f} e^{\beta H} | n_b, n_f \rangle$$

$$= \sum_{n_b=0}^{\infty} \sum_{n_f=0}^{1} (-1)^{n_f} e^{\hbar\omega\beta(n_b+n_f)} \underbrace{\langle n_b, n_f | n_b, n_f \rangle}_{1}$$

$$= \sum_{n_b=0}^{\infty} \sum_{n_f=0}^{1} (-1)^{n_f} e^{\hbar\omega\beta(n_b+n_f)}$$

$$= \sum_{n_b=0}^{\infty} \left[(-1)^0 e^{\hbar\omega\beta(n_b+0)} + (-1)^1 e^{\hbar\omega\beta(n_b+1)}\right]$$

$$= \sum_{n_b=0}^{\infty} e^{\hbar\omega\beta n_b} - \sum_{n_b=0}^{\infty} e^{\hbar\omega\beta(n_b+1)} = (use\ n_b = m-1)$$

$$= 1 + \sum_{n_b=1}^{\infty} e^{\hbar\omega\beta n_b} - \sum_{m=1}^{\infty} e^{\hbar\omega\beta m} = 1 \quad \Rightarrow \quad \text{No SUSY breaking!}$$

Problem 4: hints and solutions

Recall the rules for Grassmann integration:

$$\int db\, 1 = 0, \quad \int db\, b = 1, \quad \int db^\dagger\, 1 = 0, \quad \int db^\dagger\, b^\dagger = 1. \tag{53}$$

Witten index (Euclidean path integral approach):

$$\begin{aligned}
Z_W &= \frac{1}{2\pi i}\int da^\dagger da db^\dagger db\, e^{-\beta H} = \frac{1}{2\pi i}\int da^\dagger da db^\dagger db\, e^{-\hbar\omega\beta(n_b+n_f)} \\
&= \frac{1}{2\pi i}\int da^\dagger da\, e^{-\hbar\omega\beta a^\dagger a}\int db^\dagger db\, e^{-\hbar\omega\beta b^\dagger b} \\
&= use\ e^{-\hbar\omega\beta b^\dagger b} = 1 - \hbar\omega\beta b^\dagger b,\ b^2 = 0 = b^{\dagger 2} \\
&\Rightarrow \int db 1 \int db^\dagger 1 - \hbar\omega\beta \int db\, b \int db^\dagger\, b^\dagger = -\hbar\omega\beta \\
&= -\frac{\hbar\omega\beta}{2\pi i}\int da^\dagger da\, e^{-\hbar\omega\beta a^\dagger a} = \cdots
\end{aligned}$$

Hint: $a = x + iy$ and $a^\dagger = x - iy$.

See D. Tong for an alternative derivation:
http://www.damtp.cam.ac.uk/user/tong/susyqm.html.

Thank You!

- Partially supported by
  - The Bulgarian NSF Grants N28/5
  - The Sofia University Grant 80-10-33
  - SEENET-MTP, CERN and ICTP
- Special gratitude to G. Djordjević, D. Dimitrijević, M. Milošević, M. Stojanović, N. Konjik and Marija Dimitrijević Ćirić and all of YOU!