Introduction to Holography: Tutorials and Exercises
CERN - SEENET-MTP - ICTP SCHOOL
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1 Extra dimensions and partition functions

Problem 1.1: Consider the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\partial^2_{xx} + V(x) - E\right]\psi(x) = 0. \tag{1.1}$$

with the one-dimensional square-well potential of infinite height:

$$V(x) = \begin{cases} 0, \text{ if } x \in (0,a), \quad a > 0, \ \& \ \psi(0) = 0 = \psi(a), \\ \infty, \text{ if } x \notin (0,a) \ \Rightarrow \ \psi(x) = 0. \end{cases} \tag{1.2}$$

Determine the wave function $\psi_k(x)$ and the energy spectrum E_k, where k is the principle quantum number.

Problem 1.2 (B. Zwiebach): Add an extra circular dimension y with a small radius R to the square-well with $(x, y) = (x, y + 2\pi R)$. Now the particle moves on a cylinder with length a and circumference $2\pi R$. The potential $V(x, y)$ remains the same as in (1.2) and is y-independent.

a) Determine $\psi_{k,l}(x, y)$ and $E_{k,l}$. At what probing energies the effects of an extra dimension can be observed?

Hint: The Schrödinger equation in $2d$:

$$\left[-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y) - E\right]\psi(x,y) = 0. \tag{1.3}$$

b) Determine the statistical partition function $\mathcal{Z}(a, R)$ and show that, at high temperatures ($\beta = (kT)^{-1} \to 0$), the effects of the extra dimension are visible. Evaluate $\mathcal{Z}$ in the regime $\frac{\hbar^2}{ma^2} \leq kT \leq \frac{\hbar^2}{mR^2}$ and include the leading correction due to the small extra dimension.

2 Scalar electrodynamics

Consider a complex scalar field ϕ in $D = d + 1$ dimensions coupled to a $U(1)$ gauge field, A_μ,

$$\mathcal{L}_{kin} = -(D_\mu\phi)^* D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{2.1}$$

with $D_\mu := \partial_\mu - \mathrm{i}eA_\mu$. Determine the mass dimensions of the fields and the coupling constant e - is the theory (classically) conformal invariant? Is the theory gauge invariant? Write the equations of motion and the Feynman rules for the resulting theory (known as scalar electrodynamics). Now add the term

$$\mathcal{L}_\lambda = \lambda_1 \phi^* \phi + \lambda_2 (\phi^* \phi)^2 \ , \tag{2.2}$$

where $\lambda_{1,2}$ are real coupling constants. Is there and what is the change in the equations of motion, Feynman rules and conformal/gauge invariance after the addition of $\mathcal{L}_\lambda$? Draw your favorite 1-loop Feynman diagram obeying the rules and write down its corresponding expression with the integral over the internal momentum.

3 Classical conformal invariance

Consider the scalar field action in D-dimensions

$$\mathcal{S} = \int \mathrm{d}^D x \left(-\frac{1}{2}(\partial_\mu \phi)^2 - g\phi^3 - \lambda\phi^4 - k\phi^6 \right) . \tag{3.1}$$

Determine which of the coupling constants $\{g, \lambda, k\}$ is allowed to be non-vanishing in a classically conformally invariant theory for different values of D. Which are the special values of D allowing for interacting classically conformal theories?

4 Euclidean path integral in quantum mechanics

In quantum field theory, the Euclidean path integral on a compact time direction of period β has the interpretation of the statistical/thermal partition function Z_β if bosons are identified periodically and fermions anti-periodically. Here we change the boundary conditions and make all fields periodic (still, bosons are commuting and fermions anti-commuting), which corresponds to a different sort of observable partition function, a supersymmetric index Z_W also known as the Witten index.

Consider the double harmonic oscillator

$$H = \hbar\omega(a^\dagger a + b^\dagger b) \ , \tag{4.1}$$

which can be interpreted as a coupled bosonic and fermionic oscillator with corresponding particle numbers $n_b \in \mathbb{Z}_+$ and $n_f \in \{0, 1\}$. The eigenstates of the Hamiltonian are therefore $|n_f, n_b\rangle$ with eigenvalues (energies) $\hbar\omega(n_b + n_f)$. Keeping in mind that the operators $a, a^\dagger$ are bosonic, while $b, b^\dagger$ are fermionic, evaluate the corresponding Euclidean path integral

$$Z_W = \int \frac{\mathrm{d}a^\dagger \mathrm{d}a}{2\pi i} \mathrm{d}b^\dagger \mathrm{d}b \ e^{-\beta H} \ . \tag{4.2}$$

This path integral is a way of evaluating the Witten index and has the interpretation of counting the invariant quantity of bosonic minus fermionic states,

$$Z_W = Tr(-1)^{n_f} e^{-\beta H} \ . \tag{4.3}$$

Based on the explicit knowledge of the spectrum of the double harmonic oscillator, show that your result for Z_W indeed matches the expectation derived from the path integral. How does the final answer depend on β and why?

5 Euclidean conformal group

Determine the symmetry group of a D-dimensional Euclidean conformal field theory by Wick rotating from $(1, D-1)$ signature of the spacetime.

6 AdS coordinates

Derive explicitly the AdS_5 metric in global and Poincaré coordinates starting from the hyperboloid

$$x_0^2 + x_5^2 - \sum_{i=1}^{4} x_i^2 = L^2 \ , \tag{6.1}$$

with the metric

$$\mathrm{d}s_5^2 = -\mathrm{d}x_0^2 - \mathrm{d}x_5^2 + \sum_{i=1}^{4} \mathrm{d}x_i^2 \ . \tag{6.2}$$

7 Scalar field in AdS

Consider the action of a real scalar field on AdS_5 space,

$$\mathcal{S} = \int \mathrm{d}^5x \sqrt{-g} \left(-\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m \phi^2 - \lambda \phi^3 \right) , \tag{7.1}$$

neglecting backreaction (light scalar approximation). Derive the equation of motion and the Feynman rules. The scalar field ϕ is dual to an operator $\mathcal{O}$ in the dual conformal field theory on the asymptotic boundary. Determine whether the four-point correlator $< \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} >$ is non-vanishing by looking at the dual ϕ-correlator.

8 Scalar field on a circle

Consider the action of a real scalar field in two spacetime dimensions,

$$\mathcal{S} = \int \mathrm{d}t \ \mathrm{d}x \left(-\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} \lambda_1 \phi^2 - \lambda_2 \phi^4 \right) , \tag{8.1}$$

and assume that the spatial coordinate x parameterize a circle of length $L = 2\pi R$. Now decompose the scalar field in terms of a Fourier sum,

$$\phi(t,x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \phi_k(t) e^{ikx/R} \ , \tag{8.2}$$

where k is an integer. What condition do the ϕ_k satisfy?

Write the action in terms of ϕ_k and show that you obtain an infinite tower of harmonic oscillators with frequencies (masses)

$$M_k^2 = \lambda_1 + \frac{k^2}{R^2} \ . \tag{8.3}$$

Write down the Feynman rules, the propagators and vertices in momentum space, for this quantum-mechanical model. Draw the Feynman diagrams that contribute to the self-energy (two external lines) of ϕ_0 in the one-loop approximation, and evaluate the self-energy and therefore the correction to the mass of ϕ_0. Which corrections come from self-interactions between ϕ_0 and itself and which from interactions with the other fields $\phi_{k\neq 0}$? Now give the mass of ϕ_0 in the two opposite limits for the circle radius, $R \to 0$ and $R \to \infty$. Explain the physical meaning of the results, are they in agreement with your physical expectations?

9 Feynman rules

Consider the following set of Feynman rules (in momentum space) for the two real scalar fields ϕ and σ and a spinorial field ψ:

- free scalar propagators

$$\Delta_\phi(k) = \frac{1}{k^2 + M^2 + i\varepsilon} \ , \qquad \Delta_\sigma(k) = \frac{1}{N^2} \ . \tag{9.1}$$

- free fermion propagator

$$\Delta_\psi(k) = \frac{(-i\not{k} + m)}{k^2 + m^2 + i\varepsilon} \ . \tag{9.2}$$

- a three point vertex with one σ and two fermion legs (ψ and $\bar{\psi}$) with coefficient $(-i\ g_1)$
- a three point vertex with two ϕ and one σ legs with coefficient $(-i\ g_2)$
- a four point vertex with four ϕ legs with a factor $(-i\ \lambda)$

with coupling constants M, N, m, g, λ.

- Write down the original Lagrangian that corresponds to the Feynman rules above and derive the corresponding equations of motion for the three different fields.
- Which of the fields does not give rise to a physical particle? Pick this particular field and integrate it out of the path integral by substituting its classical solution in the Lagrangian. Write down the resulting new Lagrangian.
- Derive the Feynman rules for this new theory in the usual limit for small coupling constants.
- Write down the Feynman diagrams and their corresponding expressions (without solving the integrals) contributing to the self-energy of the fermionic field in the one-loop approximation.

10 Scalar propagator

You are given the general expression for a full two-point correlation function of a real scalar field ϕ

$$< \phi\phi >= D_\phi(p) = \frac{1}{R(p) + i\ I(p)} \ , \tag{10.1}$$

for unspecified real functions $R(p)$ and $I(p)$ of the particle momentum p. Analyze the propagator depending on the various possibilities for the two functions - in which cases the excitations of the field ϕ give rise to a physical particle and is it stable or decaying with time?

11 Gauge fixing for massive vector fields

Consider a vector field A_μ coupled to a real scalar field ϕ and a spinor field ψ in 4 spacetime dimensions, given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{M}{2\ q^2}\left|(\partial_\mu - iqA_\mu)e^{iq\phi/M}\right|^2 - \bar{\psi}(\not{\partial} - ig\ \gamma^\mu A_\mu + m)\psi\ . \tag{11.1}$$

- Show that the Lagrangian is invariant under the combined gauge transformations,

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \xi \ , \\ \phi &\to \phi + M\ \xi \ , \\ \psi &\to \exp[ig\xi]\ \psi \ , \end{aligned} \tag{11.2}$$

  where $\xi(x)$ is an arbitrary function of space and time.

- Collect all fields quadratic in the fields A_μ and ϕ. Argue that the inverse propagator takes the form of a 5×5 matrix and determine this matrix in momentum space. Does the propagator exist? Motivate your answer in two different ways: both on the basis of the explicit matrix and on the basis of a more general argument.

- Argue that $\phi = 0$ is an admissible gauge condition. Determine now the propagator for A_μ in this gauge. What are the physical bosonic states described by the resulting Lagrangian? Determine the mass dimension of all the fields and coupling constants of the theory in this gauge. Is the theory in this gauge renormalizable by power counting or not?

- Instead of the gauge $\phi = 0$ we now choose another gauge condition by adding the following term to the original Lagrangian

$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2}(\lambda\ \partial_\mu A^\mu + M\ \lambda^{-1}\phi)^2\ , \tag{11.3}$$

  with an arbitrary parameter λ. Calculate again the propagators for A_μ and ϕ. What are in this case the physical bosonic states described by the resulting Lagrangian? Determine again the mass dimensions of the fields and coupling constants of the theory with this new gauge fixing. Is the theory in this new gauge renormalizable by power counting or not?

- Write down the Feynman rules in the two gauges. Consider the one-loop fermion self-energy diagrams in the two gauges on the mass shell (i.e. sandwiched with spinors that satisfy the Dirac equation $(i\ \not{p} + m)u = 0$ that also implies $p^2 + m^2 = 0$). Determine the difference in the self-energy expression between the two gauges. Did you expect the result and why?